EE 131A
Probability
Instructor: Lara Dolecek

Winter 2020 Midterm A
Monday, February 3, 2020

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name: Solution

Your ID Number:

Name of person on your left:

Name of person on your right:

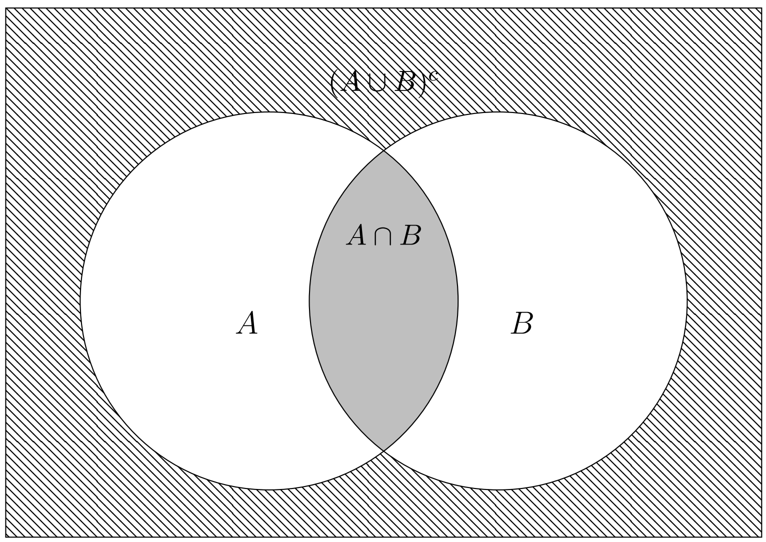
Problem	Score	Possible
1		10
2		15
3		15
4		15
5		15
6		15
7		15
Total		100

1. (5+5 pts)

   (a) Draw a Venn Diagram for the events A and B. Shade in the area corresponding to the events $(A \cap B)$ and $(A \cup B)^c$. Clearly indicate which area belongs to which event.

   (b) Assume that we throw a single six-sided die. Let A be the event that the result is even and let B be the event that the result is less than or equal to 3. What is $P(A \cap B)$ and $P(A \cup B)$?

Solution:

(a) 

(b) There are 6 equally possible outcomes for the die toss: $\{1, 2, 3, 4, 5, 6\}$. Event A corresponds to the samples $\{2, 4, 6\}$ and event B corresponds to the samples $\{1, 2, 3\}$. As such,

$$P(A \cap B) = P(\{2\}) = \frac{1}{6} \tag{1}$$

$$P(A \cup B) = P(\{1, 2, 3, 4, 6\}) = \frac{5}{6} \tag{2}$$

2. (15 pts) Let A and B be two events. Given that $P(B) > 0$, prove

$$P(A \cup B^c | B) = P(A \cap B | B).$$

You may use any result taught in lecture or homework.

Solution:

By Bayes Rule, we have

$$P(A \cup B^c | B) = \frac{P((A \cup B^c) \cap B)}{P(B)} = \frac{P((A \cap B) \cup (B^c \cap B))}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

since $B^c \cap B = \emptyset$.

We also have

$$P(A \cap B | B) = \frac{P(A \cap B \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)}.$$

Hence, the two probabilities are equal.

3. (15 pts) True or False.

Circling the correct answer is worth $+3$ points, circling the incorrect answer is worth -1 points. Not circling either is worth 0 points.

(a) For random variable X, $Var(aX) = aVar(X)$ for all a.

TRUE **FALSE**

(b) $P(A|B) = P(B|A)$ holds if and only if the events A and B are mutually exclusive.

TRUE **FALSE**

(c) A discrete random variable has jump discontinuities in its cumulative distribution function.

TRUE FALSE

(d) If events X and Y are mutually exclusive, then they are also independent.

TRUE **FALSE**

(e) For random variables X and Y, if $E[XY] = E[X]E[Y]$, then X and Y are uncorrelated.

TRUE FALSE

4. (3+5+7 pts) Assume there are 5 jars numbered 1 to 5. The ith jar contains i black balls, $6 - i$ red balls, and 5 green balls. A jar is selected uniformly at random and a ball is selected from that jar. Let the events B, R, and G represent the events that a black, red, or green ball is chosen, respectively. Let A_k represent the event that the kth jar is chosen.

   (a) What is $P(B|A_k)$?

   (b) What is $P(G)$, $P(B)$, and $P(R)$?

   (c) Given that the ball selected was black, what is the probability that the ball came from the kth jar, i.e. $P(A_k|B)$?

Solution:

(a) By the problem definition, $P(B|A_k) = \frac{k}{11}$.

(b) Regardless of which jar is chosen, the green balls always make up 5 of the 11 available balls. Hence, $P(G) = \frac{5}{11}$.
By symmetry, $P(B) = P(R)$. Therefore,

$$1 = P(B) + P(R) + P(G) = 2 \cdot P(B) + \frac{5}{11}$$
$$\implies P(B) = \frac{3}{11}$$

Hence, $P(B) = P(R) = \frac{3}{11}$.

(c) By Bayes rule, we have

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)}$$

By using the values determined in previous parts, we get

$$P(A_k|B) = \frac{\frac{k}{11} \cdot \frac{1}{5}}{\frac{3}{11}} = \frac{k}{15}$$

5. (15 pts) We select two distinct numbers (a, b) in the range 1 to 99 (inclusive). How many ways can we pick a and b such that their sum is even and a is a multiple of 9?

Solution:

This question was taken verbatim from HW1.

First, we note that there are 11 multiples of 9 in the range 1 to 99. 5 of them are even and 6 are odd. Similarly, we note that there are 50 odd numbers and 49 even numbers in the range.

To have a sum be even, the summands must either both be odd or both be even. Since a and b are distinct, we can select a first and then select b. For a and b to both be odd, there are 6 choices for a and then 49 choices for b. Similarly, for a and b to both be even, there are 5 choices for a and then 48 choices for b.

As such, the number of selections that satisfy the criteria is $6 \cdot 49 + 5 \cdot 48 = 534$.

6. (7+8 pts) Let $X_1, X_2, \ldots, X_n$ be Bernoulli random variables each with parameter p_1 for even i and $p_2 \neq p_1$ for odd i, where n is an even positive number. Let Y be the sum of all the X_i's.

   (a) Is Y a Binomial Random Variable? Justify your answer.

   (b) Compute E[Y].

Solution:

(a) No, Y is not a Binomial Random Variable as it is not a sum of independent and identically distributed Bernoulli random variables. As such, the PMF of Y can be written as

$$P(Y = k) = \sum_{i=0}^{k} \binom{\frac{n}{2}}{i} p_1^i (1 - p_1)^{\frac{n}{2} - i} \binom{\frac{n}{2}}{k - i} p_2^{k-i} (1 - p_2)^{\frac{n}{2} - (k-i)}$$

which is not a PMF of a Binomial RV for $p_1 \neq p_2$.

(b) Since $Y = X_1 + X_2 + \ldots + X_n$, we can compute the expectation of Y using the linearity of expectation as follows:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \ldots + X_n] \\ &= (E[X_1] + E[X_3] + \ldots) + (E[X_2] + E[X_4] + \ldots) \\ &= \frac{n}{2} p_1 + \frac{n}{2} p_2 \\ &= \frac{n}{2}(p_1 + p_2) \end{aligned}$$

7. (3+7+5 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} cx^2 & |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find E$[X]$ and VAR(X).

(c) Find $P(X \geq \frac{1}{2})$.

Solution:

(a) We should have

$$\int_{-1}^{1} cx^2 dx = c[\frac{1}{3}x^3]_{-1}^{1} = \frac{2}{3}c = 1,$$

so $c = \frac{3}{2}$.

(b) By symmetry of the pdf across the y axis, $E(X) = 0$.

Since $E(X) = 0$,

$$Var(X) = E(X^2) = \int_{-1}^{1} \frac{3}{2}x^4 dx = \frac{3}{2}[\frac{1}{5}x^5]_{-1}^{1} = \frac{3}{5}.$$

(c)

$$P(X \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{1} \frac{3}{2}x^2 dx = \frac{3}{2}[\frac{1}{3}x^3]_{\frac{1}{2}}^{1} = \frac{7}{16}.$$