EE 131A
Probability
Instructor: Lara Dolecek

Winter 2018 Midterm
Tuesday, February 12, 2018

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:

Your ID Number:

Name of person on your left:

Name of person on your right:

Problem	Score	Possible
1	10	10
2	10	10
3	13	15
4	15	15
5	8	10
6	10	10
7	15	15
8	12	15
Total	93	100

9/10

1. (10 pts) Show that if $P(A) > 0$, then

$$P(A \cap B|A) \geq P(A \cap B|A \cup B)$$

$$P(A \cap B | A) = \frac{P((A\cap B)\cap A)}{P(A)} \quad \text{if } P(A)>0$$

$$= \frac{P(A\cap B)}{P(A)}$$

$$\geq \frac{P(A\cap B)}{P(A)+P(A^c\cap B)} \quad \left(\text{since } P(A) \leq P(A)+P(A^c\cap B)\right)$$

not true

$$\geq \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)}$$

should be equal

$\left(\text{since } P(A\cap B) = P((A\cap B)\cap(A\cup B))\right)$

$\left(\text{since } P(A\cup B) = P(A)+P(A^c\cap B)\right)$

$$\boxed{P(A\cap B|A) \geq P(A\cap B|A\cup B)}$$

~~9~~/10

10

2. (4+6 pts) Let X be a random variable that takes integer values from 0 to 9 with equal probability $\frac{1}{10}$.

   (a) Find the PMF of the random variable $Y = X \bmod 3$.

   (b) Find the PMF of the random variable $Y = 5 \bmod (X+1)$.

a. $P(Y=0) = P(X \in \{0,3,6,9\}) = \frac{4}{10} = \frac{2}{5}$

$P(Y=1) = P(X \in \{1,4,7\}) = \frac{3}{10}$

$P(Y=2) = P(X \in \{2,5,8\}) = \frac{3}{10}$

y	P(Y=y)
0	2/5
1	3/10
2	3/10

b.

$5 \% 1 = 0$

$5 \% 2 = 1$

$5 \% 3 = 2$

$5 \% 4 = 1$

$5 \% 5 = 0$

$5 \% 6 = 5$

$5 \% 7 = 5$

$5 \% 8 = 5$

$5 \% 9 = 5$

$5 \% 10 = 5$

$P(Y=0) = P(X \in \{0,4\}) = \frac{2}{10} = \frac{1}{5}$

$P(Y=1) = P(X \in \{1,3\}) = \frac{2}{10} = \frac{1}{5}$

$P(Y=2) = P(X=2) = \frac{1}{10}$

$P(Y=5) = P(X \in \{5,6,7,8,9\}) = \frac{5}{10} = \frac{1}{2}$

y	P(Y=y)
0	1/5
1	1/5
2	1/10
5	1/2

+10

13

3. (7+8 pts)

(a) A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

(b) A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

a. multinomial: $\boxed{\frac{10!}{5!\,2!\,3!} \text{ different divisions}}$ +7

b. $P = \frac{\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{48}{1}}{\binom{52}{5}} = \boxed{\frac{13^4 \cdot 48}{\binom{52}{5}}}$ +6

$4 \quad \frac{13!}{2!\,11!} = \quad (13 \cdot 12)\ 2$

15

4. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) The expected value of a sum of random variables is equal to the sum of the expected values of each random variable.

TRUE FALSE

(b) Discrete variables have means that are always integer values.

TRUE FALSE

(c) The probability of the success of a trial or observation for a binomial probability distribution depends on the trial or observation that came before it.

TRUE FALSE

(d) If events X and Y are independent, then they are also mutually exclusive.

TRUE FALSE

(e) If events X and Y are independent, $Var[X] = a$, and $Var[Y] = b$, then $Var[a + b] = Var[X] + Var[Y]$ is always true.

TRUE FALSE VAR(a+b) = 0

8

5. (5+5 pts)

(a) Prove the memoryless property of geometric random variables.

(b) The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If David bought a functional radio which has been used for 8 years, what is the probability that it will be working after an additional 8 years?

a. Prove $P[M \geq k+j \mid M \geq j+1] = P[M \geq k] \quad \forall j,k > 1 \quad M \sim \text{geometric}(p)$

$$P[M \geq k+j \mid M \geq j+1] = \frac{P[(M \geq k+j) \cap (M \geq j+1)]}{P[M \geq j+1]}$$

$$= \frac{P[M \geq k+j]}{P[M \geq j+1]}$$

(since if $M \geq k+j$ and $k > 1$ then $M \geq j+1$ must hold true)

$$= \frac{(1-p)^{k+j-1}}{(1-p)^{j}}$$

$$= (1-p)^{k-1}$$

$$\boxed{\therefore P[M \geq k+j \mid M \geq j+1] = P[M \geq k]}$$

$$P[M \geq n] = \sum_{k=n}^{\infty} p(1-p)^{k-1}$$

$$= p\sum_{k=n-1}^{\infty} (1-p)^{k}$$

$$= p\sum_{k=0}^{\infty} (1-p)^{k} - p\sum_{k=0}^{n-2} (1-p)^{k}$$

$$= p\left[\frac{1}{1-(1-p)} - \frac{1-(1-p)^{n-1}}{1-(1-p)}\right]$$

$$= 1 - (1-(1-p)^{n-1})$$

$$P[M \geq n] = (1-p)^{n-1}$$

+5

b. Exponential is memoryless.

Let $X \sim \exp(\lambda = 1/8)$ # of years the radio functions for

$$P[X \geq 16 \text{ years} \mid X \geq 9 \text{ years}] = P[X \geq 8 \text{ years}]$$

$$P[X \geq 8 \text{ years}] = F_X(8)$$

$$= 1 - e^{-\frac{1}{8}(8)}$$

$$= \boxed{1-e^{-1}}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-\lambda x} & x \geq 0 \end{cases}$$

+3

10

6. (3+3+4 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find the expected value of X.

(c) Find the variance of X.

a. $1 = \int_{-\infty}^{\infty} f(x)\, dx$

$1 = c\int_{-1}^{1} 1 - x^2 \, dx$

$1 = c\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$

$1 = c\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$

$1 = c\,\frac{4}{3}$

$\boxed{c = \frac{3}{4}}$

b. $\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \int_{-1}^{1} x\,\frac{3}{4}(1-x^2)\, dx$

$= \frac{3}{4}\int_{-1}^{1} x - x^3\, dx$

$= \frac{3}{4}\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^{1}$

$\boxed{\mathbb{E}[X] = 0}$

c. $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_{-1}^{1} x^2\,\frac{3}{4}(1-x^2)\, dx$

$= \frac{3}{4}\int_{-1}^{1} x^2 - x^4\, dx$

$= \frac{3}{4}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{1}$

$= \frac{3}{4}\,2\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{3}{2}\,\frac{2}{15}$

$\mathbb{E}[X^2] = \frac{1}{5}$

$\mathrm{VAR}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$= \frac{1}{5} - (0)^2$

$\boxed{\mathrm{VAR}(X) = \frac{1}{5}}$

15

7. (7+8 pts)

(a) Find the characteristic function of the uniform continuous random variable, distributed uniformly on the interval $[-b, b]$.

(b) Find the mean and variance of X by applying the moment theorem.

a. $$f_x(x) = \begin{cases} \frac{1}{2b} & -b \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_x(\omega) = \mathbb{E}[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

$$= \int_{-b}^{b} e^{j\omega x} \frac{1}{2b}\, dx$$

$$= \frac{1}{2b} \int_{-b}^{b} e^{j\omega x}\, dx$$

$$= \frac{1}{2bj\omega} e^{j\omega x} \Big|_{-b}^{b}$$

$$= \frac{1}{b\omega} \frac{e^{j\omega b} - e^{-j\omega b}}{2j}$$

$$\boxed{\Phi_x(\omega) = \frac{\sin(b\omega)}{b\omega}}$$

b. $$\mathbb{E}[X] = \mathbb{E}[X^1] = \frac{1}{j} \frac{d}{d\omega} \Phi_x(\omega) \Big|_{\omega=0} = \frac{1}{j} \left[\frac{\cos(b\omega) b^2 \omega - \sin(b\omega) b}{b^2 \omega^2} \right]_{\omega=0}$$

$\cos(b\omega) \approx 1 - \frac{1}{2}(b\omega)^2$

$\sin(b\omega) \approx b\omega$

$$\approx \frac{1}{j} \left[\frac{(1 - \frac{1}{2}(b\omega)^2) b^2 \omega - b^2 \omega}{b^2 \omega^2} \right]_{\omega=0}$$

$$\approx \frac{1}{j} \left[\frac{-\frac{1}{2} b^4 \omega^3}{b^2 \omega^2} \right]_{\omega=0}$$

$\frac{(2b)^2}{12} = \frac{4b^2}{12} = \frac{b^2}{3}$ $\quad \boxed{\mathbb{E}[X] = 0}$

c. $$\mathbb{E}[X^2] = \frac{1}{j^2} \frac{d^2}{d\omega^2} \Phi_x(\omega) \Big|_{\omega=0} = - \left[\frac{[\cos(b\omega) b^2 - \sin(b\omega) b^3 \omega - \cos(b\omega) b^2] b^2 \omega^2 - [\cos(b\omega) b^2 \omega - \sin(b\omega) b] 2b^2 \omega}{b^4 \omega^4} \right]_{\omega=0}$$

$$= \frac{\sin(b\omega) b^5 \omega^3 + 2\cos(b\omega) b^4 \omega^2 - 2\sin(b\omega) b^3 \omega}{b^4 \omega^4} \Big|_{\omega=0}$$

CONTINUED ON BACK!

$$\approx \left.\frac{(bw)\,b^5w^3 + 2\left(1-\frac{1}{2}(bw)^2\right)b^4w^2 - 2(bw)\,b^3w}{b^4w^4}\right|_{w=0}$$

$$\approx \left.\frac{b^6w^4 + 2b^4w^2 - b^6w^4 - 2b^4w^2}{b^4w^4}\right|_{w=0}$$

$$\approx \left.\frac{\frac{1}{3}b^6\cancel{w^4}}{b^4\cancel{w^4}}\right|_{w=0}$$

$$\mathbb{E}[x^2] \approx \frac{b^2}{3}$$

$$\mathrm{VAR}(x) = \mathbb{E}[x^2] - (\mathbb{E}[x])^2$$

$$= \frac{b^2}{3} - (0)^2$$

$$\boxed{\mathrm{VAR}(x) = \frac{b^2}{3}}$$

12

8. (6+5+4 pts) Consider a biased coin with p being the probability of heads. We flip the coin until r tails have appeared, and then stop flipping the coin. Let X be the random variable denoting the number of heads in this experiment.

(a) Find the PMF of X.

(b) Find the expected value of X.

(c) Find the variance of X.

a. Let $Y_i \sim \text{geometric}(1-p)$. Y_i represents # of tosses after the $(i-1)^{th}$ tail until the i^{th} tail appears

$$X = (Y_1 - 1) + (Y_2 - 1) + \cdots + (Y_r - 1)$$

$$X = Y_1 + Y_2 + \cdots + Y_r - r$$

good way understanding.

+3

b. $$\mathbb{E}[X] = \mathbb{E}[Y_1 + Y_2 + \cdots + Y_r - r]$$

$$= \mathbb{E}[Y_1] + \mathbb{E}[Y_2] + \cdots + \mathbb{E}[Y_r] - r$$

$$= \frac{1}{1-p} + \frac{1}{1-p} + \cdots + \frac{1}{1-p} - r$$

$$= \frac{r}{1-p} - r$$

$$\boxed{\mathbb{E}[X] = \frac{rp}{1-p}}$$

$$\mathbb{E}[Y_i] = \frac{1}{1-p} \quad \forall i$$

c. $$VAR(Y_i) = \frac{1-(1-p)}{(1-p)^2} = \frac{p}{(1-p)^2}$$

$Y_1, Y_2, \ldots, Y_r$ are iid:

$$VAR(X) = VAR(Y_1 + Y_2 + \cdots + Y_r - r)$$

$$= VAR(Y_1) + VAR(Y_2) + \cdots + VAR(Y_r)$$

$$= r\,VAR(Y_i)$$

$$\boxed{VAR(X) = \frac{rp}{(1-p)^2}}$$

+9