EE 131A
Probability
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Winter 2018 Midterm
Tuesday, February 12, 2018

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name: Solution

Your ID Number:

Name of person on your left:

Name of person on your right:

Problem	Score	Possible
1		10
2		10
3		15
4		15
5		10
6		10
7		15
8		15
Total		100

1. (10 pts) Show that if $P(A) > 0$, then

$$P(A \cap B|A) \geq P(A \cap B|A \cup B)$$

Solution:

We have

$$P(A \cap B|A) = \frac{P(A \cap B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}.$$

and

$$P(A \cap B|A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}.$$

But since $A \cup B \supset A$, the probabilities $P(A \cup B) \geq P(A)$, so

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)}$$

giving

$$P(A \cap B|A) \geq P(A \cap B|A \cup B),$$

the desired result.

2. (4+6 pts) Let X be a random variable that takes integer values from 0 to 9 with equal probability $\frac{1}{10}$.

   (a) Find the PMF of the random variable $Y = X \bmod 3$.

   (b) Find the PMF of the random variable $Y = 5 \bmod (X + 1)$.

Solution:

(a) Using the formula $P_Y(y) = \sum_{\{x | x \bmod 3 = y\}} p_X(x)$, we obtain

$$\begin{aligned}
P_Y(0) &= P_X(0) + P_X(3) + P_X(6) + P_X(9) = \frac{4}{10}, \\
P_Y(1) &= P_X(1) + P_X(4) + P_X(7) = \frac{3}{10}, \\
P_Y(2) &= P_X(2) + P_X(5) + P_X(8) = \frac{3}{10}, \\
P_Y(y) &= 0, \quad \text{if } y \notin \{0, 1, 2\}.
\end{aligned}$$

So

$$P_Y(y) = \begin{cases} \frac{4}{10} & y = 0 \\ \frac{3}{10} & y = 1 \\ \frac{3}{10} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Similarly, using the formula $P_Y(y) = \sum_{\{x | 5 \bmod (x+1) = y\}} p_X(x)$, we obtain

$$\begin{aligned}
P_Y(0) &= P_X(4) + P_X(0) = \frac{2}{10}, \\
P_Y(1) &= P_X(3) + P_X(1) = \frac{2}{10}, \\
P_Y(2) &= P_X(2) = \frac{1}{10}, \\
P_Y(5) &= P_X(5) + P_X(6) + P_X(7) + P_X(8) + P_X(9) = \frac{5}{10}.
\end{aligned}$$

So

$$P_Y(y) = \begin{cases} \frac{2}{10} & y = 0 \\ \frac{2}{10} & y = 1 \\ \frac{1}{10} & y = 2 \\ \frac{5}{10} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

3. (7+8 pts)

   (a) A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

   (b) A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

Solution:

(a) There are $\frac{10!}{5!2!3!} = 2520$ possible divisions (using multinomial coefficients).
(b) We have $\binom{52}{5}$ possible five card hands from our fifty-two cards. To have one card from each of the four suits we need to count the number of ways to select one club from the thirteen available, (this can be done in $\binom{13}{1}$ ways) one spade from the thirteen available, (this can be done in $\binom{13}{1}$ ways), one heart from the thirteen available in $\binom{13}{1}$ ways etc. The last card can be selected in $\binom{52-4}{1}$ ways. Thus we have $\binom{13}{1}^4\binom{48}{1}$ possible hands containing one card from each suit, where the order of the choice made in the $\binom{48}{1}$ selections and the corresponding selection from the $\binom{13}{1}$ that has a suit that matches the $\binom{48}{1}$ selection mater. To better explain this say when picking clubs we get the three card. When we pick from the 48 remaining cards (after having selected a card of each suit) assume we select a four of clubs. This hand is equivalent to having picked the four of clubs first and then the three of clubs. So we must divide the above by a 2! giving a probability of

$$\frac{\frac{1}{2!}\binom{13}{1}^4\binom{48}{1}}{\binom{52}{5}} = 0.2637.$$

4. (15 pts) True or False.

Circling the correct answer is worth $+3$ points, circling the incorrect answer is worth -1 points. Not circling either is worth 0 points.

(a) The expected value of a sum of random variables is equal to the sum of the expected values of each random variable.

TRUE FALSE

(b) Discrete variables have means that are always integer values.

TRUE **FALSE**

(c) The probability of the success of a trial or observation for a binomial probability distribution depends on the trial or observation that came before it.

TRUE **FALSE**

(d) If events X and Y are independent, then they are also mutually exclusive.

TRUE **FALSE**

(e) If events X and Y are independent, $Var[X] = a$, and $Var[Y] = b$, then $Var[a + b] = Var[X] + Var[Y]$ is always true.

TRUE **FALSE**

5. (5+5 pts)

(a) Prove the memoryless property of geometric random variables.

(b) The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If David bought a functional radio which has been used for 8 years, what is the probability that it will be working after an additional 8 years?

Solution:

(a) **Proof:**

Let M be a geometric random variable. Then

$P[M = k] = (1-p)^{k-1}p \quad k = 1, 2, ...$

The probability that $M \le k$ can be written in closed form:

$$P[M > k] = 1 - P[M \le k] = 1 - \sum_{j=1}^{k}(1-p)^{j-1}p = (1-p)^k$$

$$\begin{aligned} P[M \ge k+j | M > j] &= \frac{P[M \ge k+j, M > j]}{P[M > j]} = \frac{P[M \ge k+j]}{P[M > j]} \text{ for } k \ge 1 \\ &= \frac{(1-p)^{k+j-1}}{(1-p)^j} \\ &= (1-p)^{k-1} \\ &= P[M \ge k] \end{aligned}$$

Therefore, we show geometric random variable satisfies the memoryless property.

(b) Because of the memoryless property of the exponential distribution, the fact that the radio is used is irrelevant. The probability requested is then

$$P[T > 8 + t | T > t] = P[T > 8] = 1 - P[T < 8] = 1 - (1 - e^{-\frac{1}{8}8}) = e^{-1}.$$

6. (3+3+4 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} c(1-x^2) & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find the expected value of X.

(c) Find the variance of X.

Solution:

(a) We should have

$$\int_{-1}^{1} c(1-x^2)dx = c[x - \frac{1}{3}x^3]_{-1}^{1} = \frac{4}{3}c = 1,$$

so $c = \frac{3}{4}$.

(b) By symmetry of the pdf across the y axis, $E(X) = 0$.

(c) Since $E(X) = 0$,

$$Var(X) = E(X^2) = \int_{-1}^{1} \frac{3}{4}x^2(1-x^2)dx = \frac{3}{4}[\frac{1}{3}x^3 - \frac{1}{5}x^5]_{-1}^{1} = \frac{1}{5}.$$

7. (7+8 pts)

   (a) Find the characteristic function of the uniform continuous random variable, distributed uniformly on the interval $[-b, b]$.

   (b) Find the mean and variance of X by applying the moment theorem.

Solution:

(a) The characteristic function of a random variable X is defined as following:

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

Here, $j = \sqrt{-1}$ is the imaginary unit number.
For a uniform random variable in the interval $[-b, b]$,

$$f_X(x) = \frac{1}{b - (-b)} = \frac{1}{2b}$$

Thus, characteristic function of a uniform random variable in the interval $[-b, b]$ is,

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-b}^{b} \left(\frac{1}{2b}\right) e^{j\omega x} dx = \frac{1}{2b}\left[\frac{e^{j\omega x}}{j\omega}\right]_{-b}^{b} = \frac{e^{j\omega b} - e^{-j\omega b}}{2bj\omega} = \frac{\sin\omega b}{\omega b}$$

(b) The moment theorem states that the moments of X are given by

$$E[X^n] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_X(\omega)\Big|_{\omega=0}$$

This gives,

$$E[X] = \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)\Big|_{\omega=0} = \frac{1}{j}\frac{d}{d\omega}\left(\frac{e^{j\omega b} - e^{-j\omega b}}{2bj\omega}\right)\Big|_{\omega=0} = -\frac{1}{2b}\left(-\frac{1}{2}b^2 + \frac{1}{2}b^2\right) = -\frac{1}{2b}0 = 0$$

Hence, the required mean is, $E[X] = 0$.
Now, find the variance of X.

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{j^2}\frac{d^2}{d\omega^2}\Phi_X(\omega)\Big|_{\omega=0} - (0)^2 = \frac{1}{j^2}\frac{d^2}{d\omega^2}\frac{e^{j\omega b} - e^{-j\omega b}}{2bj\omega}|_{\omega=0} - 0$$
$$= -\frac{1}{2jb}\left(-\frac{jb^3}{3} - \frac{jb^3}{3}\right) = -\frac{1}{2jb}\left(-\frac{2jb^3}{3}\right) = \frac{b^2}{3}$$

Hence, the required variance is, $Var[X] = \frac{b^2}{3}$.

8. (6+5+4 pts) Consider a biased coin with p being the probability of heads. We flip the coin until r tails have appeared, and then stop flipping the coin. Let X be the random variable denoting the number of heads in this experiment.

   (a) Find the PMF of X.

   (b) Find the expected value of X.

   (c) Find the variance of X.

Solution:

(a) Suppose that $X = k$. Then, in the first $k + r - 1$ coin flips, there are k heads and $r - 1$ tails, and the last coin flip is tails. Since there $\binom{k+r-1}{k}$ different ways of positioning the k heads in $k + r - 1$ coin flips, the probability that $X = k$ is

$$P[X = k] = \binom{k+r-1}{k} p^k (1-p)^{r-1}(1-p) = \binom{k+r-1}{k} p^k (1-p)^r, k = 0, 1, 2, \ldots$$

(b) Since X has a valid PMF, we have

$$\sum_{k\geq 0} \binom{k+r-1}{k} p^k (1-p)^r = 1.$$

We find the expectation by taking derivative with respect topfrom both sides.

$$\sum_{k\geq 0} \binom{k+r-1}{k} p^k = (1-p)^{-r} \Rightarrow$$

$$\sum_{k\geq 0} \binom{k+r-1}{k} kp^{k-1} = r(1-p)^{-r-1} \Rightarrow$$

$$E[X] = \sum_{k\geq 0} \binom{k+r-1}{k} kp^k (1-p)^r = \frac{pr}{1-p}.$$

(c) We find the variance by taking another derivative with respect to p. By part(b), we have

$$\sum_{k\geq 0} \binom{k+r-1}{k} kp^k = pr(1-p)^{-r-1} \Rightarrow$$

$$\sum_{k\geq 0} \binom{k+r-1}{k} k^2 p^{k-1} = r[(1-p)^{-r-1} + (r+1)p(1-p)^{-r-2}] \Rightarrow$$

$$E[X^2] = \sum_{k\geq 0} \binom{k+r-1}{k} k^2 p^k (1-p)^r = rp[(1-p)^{-1} + (r+1)p(1-p)^{-2}] \Rightarrow$$

$$var[X] = E[X^2] - E[X]^2 = \frac{rp}{(1-p)^2}.$$