zicong - solutions.

EE 131A
Probability
Instructor: Lara Dolecek

Fall 2011 Midterm A
Wednesday, October 26, 2011

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:

Your ID Number:

Name of person on your left:

Name of person on your right:

Problem	Score	Possible
1		5
2		10
3		15
4		10
5		15
6		10
7		20
8		15
Total		100

1. (5 pts) Express the probability of the event "neither A nor B" occurs in terms of $P(A)$, $P(B)$ and $P(A \cap B)$.

event "neither A nor B" is $A^c \cap B^c$. thus

$$P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

2. (10 pts) Consider a 12-sided unfair die, with sides numbered by integers 1 through 12. Suppose the even numbered sides are twice as likely as odd numbered sides.

   Let $A = \{$ odd numbered side $\}$ and $B = \{4, 5, 6, 7, 8\}$.

   (a) Compute $P(A)$.

   (b) Compute $P(A \cap B)$.

(a) $p_1 = p_3 = p_5 = \cdots = p_{11} = \frac{1}{6\times2+6\times1} = \frac{1}{18}$

$p_2 = p_4 = p_6 = \cdots = p_{12} = \frac{2}{6\times2+6\times1} = \frac{1}{9}$

$P(A) = \frac{1}{18} \times 6 = \frac{1}{3}$

(b) $A \cap B = \{5, 7\}$

$P(A \cap B) = \frac{1}{18} \times 2 = \frac{1}{9}$

Midterm B:

(a) $P(A) = \frac{1}{15} \times 5 = \frac{1}{3}$

(b) $P(A \cap B) = \frac{1}{15} \times 2 = \frac{2}{15}$

3. (15 pts) Find the probability that in the class of 36 students, exactly 3 students have a birthday each month.

~~multinomial~~ r.v.

multinomial

$$p = \frac{36!}{3!3!\cdots 3!}\left(\frac{1}{12}\right)^3\left(\frac{1}{12}\right)^3\cdots\left(\frac{1}{12}\right)^3$$

$$= \frac{36!}{6^{12}\cdot 12^{36}}$$

or $$p = \frac{\binom{36}{3}\binom{33}{3}\binom{30}{3}\cdots\binom{6}{3}\binom{3}{3}}{12^{36}}$$

$$= \frac{36!}{6^{12}\cdot 12^{36}}$$

Midterm B:

$$p = \frac{48!}{24^{12}\cdot 12^{48}}$$

4. (10 pts) A biased coin is tossed $n = 10$ times, with $P(head) = p$. Let Y be a random variable denoting the differences between the number of heads and the number of tails.

Find the PMF of Y.

Denote number of heads as X then

$$Y = X - (n - X) = 2X - 10$$

$$P(X = k) = \binom{10}{k} p^{k} (1-p)^{10-k}$$

For $Y = m = -10, -8, -6, \cdots 6, 8, 10$,

$$P(Y = m) = P(2X - 10 = m) = P\left(X = \frac{10+m}{2}\right)$$

$$= \binom{10}{\frac{10+m}{2}} p^{\frac{10+m}{2}} (1-p)^{\frac{10-m}{2}}$$

Midterm B:

$$P(Y = m) = \binom{10}{\frac{10-m}{2}} p^{\frac{10-m}{2}} (1-p)^{\frac{10+m}{2}}$$

5. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) $E[X+Y] = E[X] + E[Y]$ when X and Y are independent.

TRUE FALSE

(b) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$

TRUE FALSE

(c) Geometric RV is a sum of independent Bernoulli RVs, each with success probability p.

TRUE FALSE

(d) If X is Bernoulli with success probability 0.5, and $Y = X + 100$ then $VAR(Y) = 25$.

TRUE FALSE

(e) For $0 < x < +\infty$, $Q(x) + Q(-x/2) = 1$

TRUE FALSE

Exo Midterm B.

(a) TRUE

(b) FALSE

(c) FALSE

(d) FALSE

(e) TRUE

6. (10 pts) Suppose $X \sim \mathcal{N}(-1, 2)$. Compute $P(-2 < X \leq 2)$ in terms of Q-function.

$X \sim N(-1,2)$ then $\dfrac{X-(-1)}{\sqrt{2}} \sim N(0,1)$.

$P(-2 < X \leq 2) = P(X \leq 2) - P(X \leq -2)$

$= 1 - Q(\frac{2+1}{\sqrt{2}}) - 1 + Q(\frac{-2+1}{\sqrt{2}})$

$= Q(-\frac{1}{\sqrt{2}}) - Q(\frac{3}{\sqrt{2}})$

Midterm B. $X \sim N(-2,1)$ then $X+2 \sim N(0,1)$

$P(-2 < X \leq 2) = P(X \leq 2) - P(X \leq -2)$

$= 1 - Q(\frac{2+2}{1}) - 1 + Q(\frac{-2+2}{1})$

$= Q(0) - Q(4)$

7. (5+ 5+5+5 pts) Suppose X is a continuous random variable uniformly distributed on the interval $[-1, +1]$.

(a) Compute $E[X]$ and $E[X^2]$.

(b) Suppose $Y = 3X + 2$. Sketch the CDF and PDF of Y.

(c) Compute $VAR(Y)$.

(d) Suppose Z is -1 if $X \leq 0$ and is $+1$ if $X > 0$. Sketch the CDF of Z.

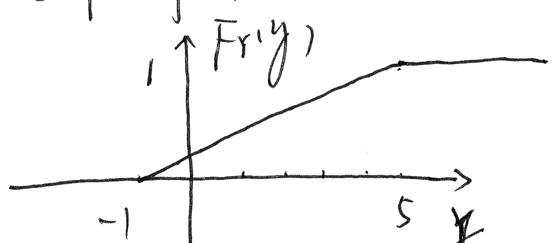
(a) $E[X] = \frac{-1+1}{2} = 0$

$E[X^2] = E[X]^2 + VAR[X] = VAR[X] = \frac{(1+1)^2}{12} = \frac{1}{3}$

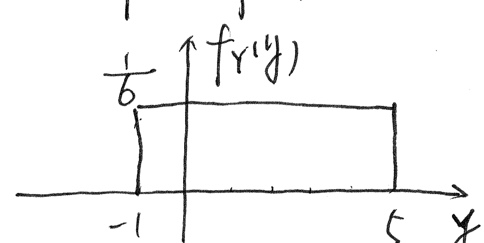
(b) $X \sim U[-1, +1]$ $Y = 3X+2$ is still uniform

then $Y \sim U[-1, 5]$

CDF of Y



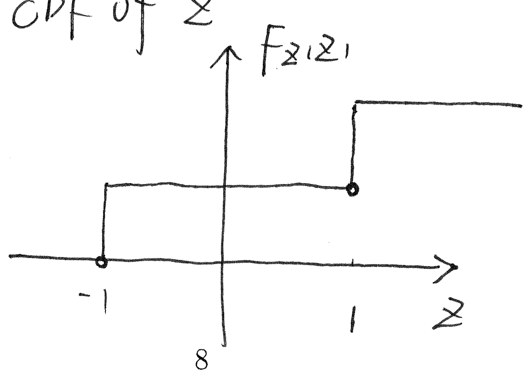
PDF of Y



(c) $VAR[Y] = VAR[3X+2] = 9VAR[X] = 3$

(d) $P(Z=-1) = \int_{-1}^{0} \frac{1}{2} dx = \frac{1}{2}$ $P(Z=1) = \int_{0}^{1} \frac{1}{2} dx = \frac{1}{2}$

CDF of Z



Midterm B.

(b) $Y \sim U[-2, 4]$

8. (10+5 pts) Suppose X is a Gaussian random variable with mean m and variance σ^2.

(a) Compute the pdf of $Y = |X|$.

(b) Is Y a continuous RV ? Why or why not ?

(a) $F_Y(y) = P(Y \le y) = P(|X| \le y)$ for $y \ge 0$

$= P(-y \le X \le y) = P(X \le y) - P(X \le -y) = F_X(y) - F_X(-y)$

$= \Phi(\frac{y-m}{\sigma}) - \Phi(\frac{-y-m}{\sigma})$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+m)^2}{2\sigma^2}}$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+m)^2}{2\sigma^2}} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Y is a continuous RV.

because pdf of Y does not have delta function, which leads discontinuity in cdf. cdf of Y should be continuous. therefore Y is a continuous R.V.