EE113: Digital Signal Processing Spring 2016

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Midterm Solutions

Problem 1:

(a) Justifications are given as follows.

Linearity: Let $x_1(n)$ and $x_2(n)$ be two distinct inputs. By applying their arbitrary linear combination $\alpha x_1(n) + \beta x_2(n)$ to the system, we can see that the output $y(n)$ is equal to $\alpha y_1(n) + \beta y_2(n)$ when $n = 5k$ for an integer k, whereas $y(n) = 0$ otherwise. Hence, the superposition principle is satisfied, and the output $y(n)$ is a linear combination of the individual outputs $y_1(n)$ and $y_2(n)$. Therefore, the system is linear.

Stability: For every bounded sequence $|x(n)| \leq B < \infty$, with B being an arbitrary finite positive number, we have that

$$|y(n)| = \begin{cases} |x\left(\frac{n}{5}\right)| \leq B < \infty, & n = 5k, k \,\text{is integer} \\ 0, & n \neq 5k, k \,\text{is integer}, \end{cases}$$

implying that $|y(n)|$ is a bounded output whenever the input is bounded.

Causality: The system is non-causal because outputs depend on futuristic inputs. For instance, $y(-5)$ depends on $x(-1)$.

Time-invariance: Consider the inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-5)$. This shows that $y(n)$ is not time-invariant since applying a shifted-version of an input $x(n)$ to the system does not result in a shifted-version of the output.

Relaxation: If the first non-zero sample of $x(n)$ is at $n = -1$, then $y(-5)$ will be non-zero, which is before -1. Thus, the system is not relaxed.

The system is linear, BIBO stable, non-causal, not time-invariant, and not relaxed.

(b) The z-transform $Y(z)$ can be computed as follows

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{5}\right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(k) z^{-5k} \\
&= \sum_{k=-\infty}^{\infty} x(k) \left(z^5\right)^{-k} \\
&= X(z^5).
\end{aligned}$$

Thus, the z-transform $Y(z)$ in terms of that of $X(z)$ is $Y(z) = X(z^5)$.

Problem 2:

(a) The zero-input response is found by figuring out the modes, and then obtaining the homogeneous solution as a linear combination of the modes as follows. The system's characteristic equation is

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0,$$

which factorizes as

$$(\lambda - \frac{1}{2})\,(\lambda - \frac{1}{3}) = 0,$$

and hence the modes are given by $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{3}$. A general form of the zero-input response is then given by

$$y_{zi}(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n.$$

Using the initial conditions $y(-2) = 1$ and $y(-1) = 2$, we can construct the following equations. This leads to the equations

$$2C_1 + 3C_2 = 2,$$

$$4C_1 + 9C_2 = 1,$$

the solutions of which is $C_1 = \frac{5}{2}$ and $C_2 = -1$.

Thus, the zero-input response of the system is $y_{zi}(n) = \frac{5}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n$, $n \geq 0$.

(b) We relax the system by forcing zero initial-conditions, which makes the system act as an LTI system, the impulse response of which is given by $y(n) = h(n)$ when the input is a unit-sample $x(n) = \delta(n)$. That is, we have that

$$h(n) - \frac{5}{6}h(n-1) + \frac{1}{6}h(n-2) = \delta(n-1).$$

We know that

$$h(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n, n \geq 2.$$

The initial conditions are obtained by substitutions as $h(0) = 0$ and $h(1) = 1$. This leads to the two equations following equations. This leads to the equations

$$C_1 + C_2 = 0,$$

$$\frac{1}{2}C_1 + \frac{1}{3}C_2 = 1,$$

the solutions of which are $C_1 = 6$ and $C_2 = -6$.

Thus, the impulse response $h(n)$ of the system is $h(n) = 6\left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right) u(n-1)$.

The zero-state response is simply the result of convolving the input $x(n) = \left(\frac{1}{4}\right)^n u(n)$ with $h(n)$ as follows

$$\begin{aligned}
y_{zs}(n) &= x(n) \star h(n) \\
&= \sum_{k=-\infty}^{\infty} h(n-k)\, x(k) \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k u(k)\; 6 \left(\left(\frac{1}{2}\right)^{n-k} - \left(\frac{1}{3}\right)^{n-k}\right) u(n-k-1) \\
&= 6 \sum_{k=0}^{n-1} \left(\frac{1}{4}\right)^k \left(\left(\frac{1}{2}\right)^{n-k} - \left(\frac{1}{3}\right)^{n-k}\right) \\
&= 6 \left(\frac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k - 6 \left(\frac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k \\
&= 6 \left(\frac{1}{2}\right)^n \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} - 6 \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}} \\
&= 12 \left(\frac{1}{2}\right)^n - 24 \left(\frac{1}{3}\right)^n + 12 \left(\frac{1}{4}\right)^n.
\end{aligned}$$

Thus, the zero-state response $y_{zs}(n)$ is given by $y_{zs}(n) = 12\left(\frac{1}{2}\right)^n - 24\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n$, $n \geq 1$.

(c) The complete response is given by

$$y_c(n) = y_{zi}(n) + y_{zs}(n),$$

thus we have that

$$y_c(n) = \frac{5}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n + 12\left(\frac{1}{2}\right)^n - 24\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n.$$

Therefore, the complete response $y_c(n)$ is given by

$$y_c(n) = \frac{29}{2}\left(\frac{1}{2}\right)^n - 25\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n, n \geq 0.$$

(d)

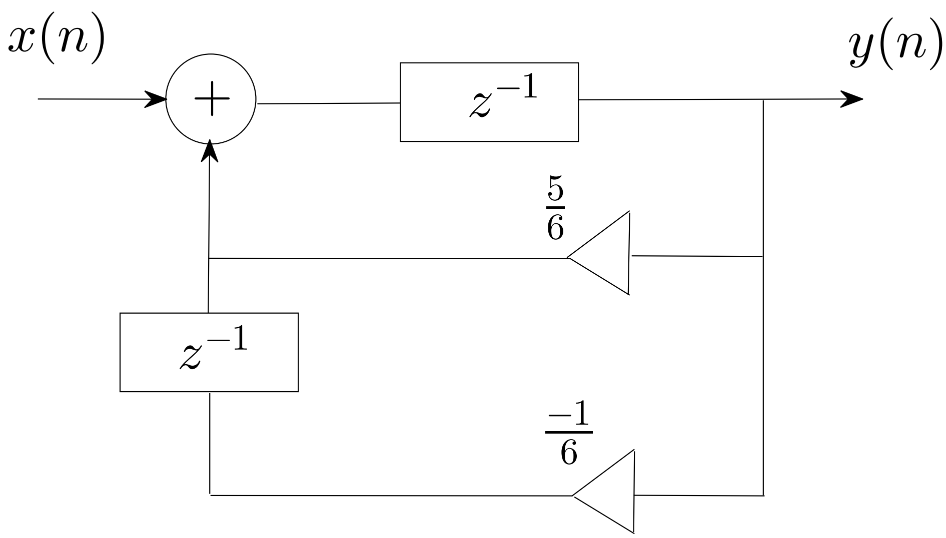

Figure 1: The block diagram for problem 2-(d).

Problem 3:

(a) The impulse-response should solve the difference equation

$$h(n) = \frac{1}{4}h(n-1) + \frac{1}{8}h(n-2) + \delta(n),$$

with the initial conditions $h(-1) = 0$ and $h(0) = 1$. The characteristic polynomial of the CCDE is given by

$$\lambda^2 - \frac{1}{4}\lambda - \frac{1}{8} = 0,$$

where

$$(\lambda - \frac{1}{2})(\lambda + \frac{1}{4}) = 0,$$

thus the modes are $\lambda = \frac{1}{2}, \frac{-1}{4}$. Thus, the impulse response is described by the homogeneous solution as

$$h(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{-1}{4}\right)^n.$$

With $h(-1) = 0$ and $h(0) = 1$, we obtain the following two equations

$$2C_1 - 4C_2 = 0,$$

$$C_1 + C_2 = 1,$$

the solution of which is $C_1 = \frac{2}{3}$ and $C_2 = \frac{1}{3}$.

Therefore, the impulse response is given by

$$h(n) = \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n, n \geq 0.$$

(b) The output is given by

$$\begin{aligned} y(n) &= x(n) \star h(n) \\ &= (\delta(n+1) - \delta(n-1)) \star h(n) \\ &= h(n+1) - h(n-1) \\ &= \left(\frac{2}{3}\left(\frac{1}{2}\right)^{n+1} + \frac{1}{3}\left(\frac{-1}{4}\right)^{n+1}\right) u(n+1) - \left(\frac{2}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{3}\left(\frac{-1}{4}\right)^{n-1}\right) u(n-1). \end{aligned}$$

Thus, for $-1 \leq n \leq 0$, we have that

$$y(n) = \frac{1}{3}\left(\frac{1}{2}\right)^n - \frac{1}{12}\left(\frac{-1}{4}\right)^n,$$

and for $n \geq 1$ we have that

$$y(n) = \frac{5}{4}\left(\frac{-1}{4}\right)^n - \left(\frac{1}{2}\right)^n.$$

Therefore, the output is given by

$$y(n) = \begin{cases} 0, & n < -1, \\ \frac{1}{3}\left(\frac{1}{2}\right)^n - \frac{1}{12}\left(\frac{-1}{4}\right)^n, & -1 \leq n \leq 0 \\ \frac{5}{4}\left(\frac{-1}{4}\right)^n - \left(\frac{1}{2}\right)^n, & n > 0, \end{cases}$$

(c) We compute the energy of the sequence for $n > 0$, if this converges, then the sequence has to be an energy sequence.

$$\begin{aligned} E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} |\alpha^n\, h(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} \alpha^{2n} \left|\frac{5}{4}\left(\frac{-1}{4}\right)^n - \left(\frac{1}{2}\right)^n\right|^2 \\ &= \sum_{n=-\infty}^{\infty} \alpha^{2n} \left(\frac{25}{16}\left(\frac{1}{16}\right)^n - \frac{5}{2}\left(\frac{-1}{8}\right)^n + \left(\frac{1}{4}\right)^n\right) \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{25}{16}\left(\frac{\alpha^2}{16}\right)^n - \frac{5}{2}\left(\frac{-\alpha^2}{8}\right)^n + \left(\frac{\alpha^2}{4}\right)^n\right). \end{aligned}$$

Therefore, for finite energy, we require that

$$\left|\frac{\alpha^2}{16}\right| < 1, \left|\frac{\alpha^2}{8}\right| < 1, \text{ and } \left|\frac{\alpha^2}{4}\right| < 1.$$

Therefore, we a finite energy sequence requires that

$$|\alpha| < \min\left\{\sqrt{16}, \sqrt{8}, \sqrt{4}\right\} = 2.$$