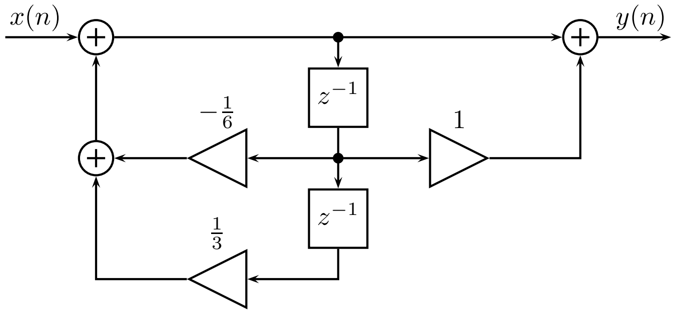
1. A causal system is described by the following block diagram.




(a) Determine the constant coefficient difference equation that describes the system, and find its impulse response.

Solution:
Interchanging the order of the two systems in cascade, we get

$$y(n) = -\frac{1}{6}y(n-1) + \frac{1}{3}y(n-2) + x(n) + x(n-1).$$

The impulse response can be found by computing the zero-state response to $x(n) = \delta(n)$. We first find the homogeneous solution:

$$\lambda^2 + \frac{1}{6}\lambda - \frac{1}{3} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = -\frac{2}{3}$$

and hence,

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(-\frac{2}{3}\right)^n, \quad \text{for all } n.$$

Letting $x(n) = \delta(n)$ and assuming that the system is relaxed, we get $y(0) = x(0) = 1$ and $y(1) = -\frac{1}{6}y(0) + x(1) = \frac{5}{6}$. From the homogeneous solution we get

$$y(0) = C_1 + C_2 = 1$$
$$y(1) = \frac{1}{2}C_1 - \frac{2}{3}C_2 = \frac{5}{6}$$

and solving for $C_1$ and $C_2$ yields $C_1 = \frac{9}{7}$ and $C_2 = -\frac{2}{7}$, and consequently,

$$h(n) = \left[\frac{9}{7}\left(\frac{1}{2}\right)^n - \frac{2}{7}\left(-\frac{2}{3}\right)^n\right]u(n).$$

(b) Given $x(n) = n2^n u(-n)$, find the output of the system using the $z$-transform.

Solution:
The output $y(n)$ when $x(n) = n2^n u(-n)$ can be found by computing the inverse $z$-transform of $Y(z) = H(z)X(z)$. The $z$-transform of $h(n)$ is readily found from the impulse response:

$$H(z) = \frac{1 + z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}}, \quad \text{ROC: } |z| > \frac{2}{3}.$$

The $z$-transform of $x(n)$ is given by

$$\begin{aligned} X(z) = \mathcal{Z}\{x(n)\} &= -z\frac{d}{dz}\mathcal{Z}\{2^n u(-n)\} = -z\frac{d}{dz}\left[\sum_{n=-\infty}^{0} 2^n z^{-n}\right] \\ &= -z\frac{d}{dz}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right] = -z\frac{d}{dz}\left[\frac{1}{1-\frac{1}{2}z}\right] = -\frac{2z}{(2-z)^2}, \quad \text{ROC: } |z| < 2. \end{aligned}$$

Consequently, $Y(z)$ is given by

$$Y(z) = H(z)X(z) = \frac{-2z^3 - 2z^2}{(z-\frac{1}{2})(z+\frac{2}{3})(z-2)^2}, \quad \text{ROC: } \frac{2}{3} < |z| < 2.$$

Using partial fraction expansion, we get

$$Y(z) = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{2}{3}} + \frac{C}{z-2} + \frac{D}{(z-2)^2}$$

where

$$\begin{aligned} A &= \left(z - \frac{1}{2}\right)Y(z)\Big|_{z=\frac{1}{2}} = -\frac{2}{7} \\ B &= \left(z + \frac{2}{3}\right)Y(z)\Big|_{z=-\frac{2}{3}} = \frac{1}{28} \\ C &= \frac{d}{dz}\left[(z-2)^2 Y(z)\right]\Big|_{z=2} = -\frac{7}{4} \\ D &= (z-2)^2 Y(z)\Big|_{z=2} = -6. \end{aligned}$$

Thus,

$$Y(z) = -\frac{2}{7}z^{-1}\frac{z}{z-\frac{1}{2}} + \frac{1}{28}z^{-1}\frac{z}{z+\frac{2}{3}} - \frac{7}{4}z^{-1}\frac{z}{z-2} - \frac{6}{2}z^{-1}\frac{2z}{(z-2)^2}$$

and since the ROC of $Y(z)$ is given by $\frac{2}{3} < |z| < 2$, we get

$$\begin{aligned} \frac{z}{z-\frac{1}{2}} &\leftrightarrow \left(\frac{1}{2}\right)^n u(n) \\ \frac{z}{z+\frac{2}{3}} &\leftrightarrow \left(-\frac{2}{3}\right)^n u(n) \\ \frac{z}{z-2} &\leftrightarrow -2^n u(-n-1) \\ \frac{2z}{(z-2)^2} &\leftrightarrow -n2^n u(-n-1) \end{aligned}$$

and consequently,

$$y(n) = \left[ -\frac{2}{7} \left( \frac{1}{2} \right)^{n-1} + \frac{1}{28} \left( -\frac{2}{3} \right)^{n-1} \right] u(n-1) + \left[ \frac{7}{4} 2^{n-1} + 3(n-1) 2^{n-1} \right] u(-n).$$

Alternatively, using partial fraction expansion on $\tilde{Y}(z) = Y(z)/z$, we would get

$$y(n) = \left[ -\frac{4}{7} \left( \frac{1}{2} \right)^{n} - \frac{3}{56} \left( -\frac{2}{3} \right)^{n} \right] u(n) + \left( -\frac{5}{8} + \frac{3}{2} n \right) 2^n u(-n-1).$$

2. The difference equation of a relaxed system is:

$$y(n) + 0.5y(n-1) - 0.14y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

Solution:
By setting $x(n) = \delta(n)$, we have

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = 0. \tag{1}$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda + \frac{7}{10}\right)\left(\lambda - \frac{1}{5}\right) = 0,$$

thus, the modes are

$$\lambda_1 = -\frac{7}{10}, \quad \lambda_2 = \frac{1}{5}.$$

Hence,

$$h(n) = C_1\left(-\frac{7}{10}\right)^n + C_2\left(\frac{1}{5}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{7}{9}, \quad C_2 = \frac{2}{9}.$$

Therefore,

$$h(n) = \left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right] u(n).$$

(b) If the input to the system is $x(n) = 2\delta(n) + \delta(n-2)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= [2\delta(n) + \delta(n-2)] * h(n) \\ &= 2h(n) + h(n-2). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \geq 2$,

$$
\begin{aligned}
y(n) &= 2h(n) + h(n-2) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) + \left[\frac{7}{9}\left(-\frac{7}{10}\right)^{n-2} + \frac{2}{9}\left(\frac{1}{5}\right)^{n-2}\right]u(n-2) \\
&= \left[2\cdot\frac{7}{9} + \frac{7}{9}\cdot\left(\frac{7}{10}\right)^{-2}\right]\left(-\frac{7}{10}\right)^n + \left[2\cdot\frac{2}{9} + \frac{2}{9}\cdot\left(\frac{1}{5}\right)^{-2}\right]\left(\frac{1}{5}\right)^n \\
&= \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n.
\end{aligned}
$$

ii) $0 \leq n \leq 1$,

$$
\begin{aligned}
y(n) &= 2h(n) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) \\
&= \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n.
\end{aligned}
$$

Therefore,

$$
y(n) = \begin{cases} 0, \quad n < 0, \\ \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n, \quad n = 0, 1, \\ \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n, \quad n \geq 2. \end{cases}
$$

(c) For what values of $\alpha$ is $g(n) = \alpha^n h(n)$ a finite energy sequence?

<u>Solution</u>:
The energy $E_g$ of $g(n)$ is given by

$$
\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty} \left|\alpha^n \left\{\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right\}\right|^2 \\
&= \sum_{n=0}^{\infty} \left|\frac{7}{9}\left(-\frac{7}{10}\alpha\right)^n + \frac{2}{9}\left(\frac{1}{5}\alpha\right)^n\right|^2 \\
&= \sum_{n=0}^{\infty} \frac{49}{81}\left(\frac{49}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{4}{81}\left(\frac{1}{25}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{28}{81}\left(-\frac{7}{50}\alpha^2\right)^n.
\end{aligned}
$$

To be a finite energy,

$$
\left|\frac{49}{100}\alpha^2\right| < 1, \quad \left|\frac{1}{25}\alpha^2\right| < 1, \text{ and } \left|\frac{7}{50}\alpha^2\right| < 1,
$$

or equivalently,

$$|\alpha| < \frac{10}{7}, \quad |\alpha| < 5, \text{ and } |\alpha| < \sqrt{\frac{50}{7}}.$$

Hence, $\alpha$ should have a value in the range of $|\alpha| < \frac{10}{7}$.

3. Consider a relaxed system with input $x[n]$ and output $y[n]$ that satisfy the difference equation

$$y[n] = ny[n-1] + x[n]$$

(a) If $x[n] = \delta[n]$ is input to the system then determine the output $y[n]$ for all $n$.

Suppose $x[n] = \delta[n]$. Since the system is relaxed we know $y[n] = 0$ for $n < 0$. For $n \geq 0$

$$y[0] = 1,\ y[1] = 1,\ y[2] = 2,\ y[3] = 6,\ y[4] = 24$$

From the above we can deduce that $y[n] = n!u[n]$

(b) Is the system Linear? Justify your answer.

Consider the following sequences $x_1[n] = a\delta[n]$ and $x_2[n] = b\delta[n]$ and let the corresponding outputs be $y_1[n]$ and $y_2[n]$ respectively. Check that $y_1[n] = an!u[n]$ and $y_2[n] = bn!u[n]$. If $x[n] = x_1[n] + x_2[n]$ then if the system is linear then the output has to be $y_1[n] + y_2[n]$. The output corresponding to $x[n]$ is denoted as $y[n]$ and is computed as follows.

For $n < 0$ $y[n] = 0$, this is true because the system is relaxed.

$$y[0] = a + b,\ y[1] = a + b,\ y[2] = 2(a + b),\ y[3] = 6(a + b),\ y[4] = 24(a + b)..$$

Hence, from the above $y[n] = n!u[n](a + b)$, which is equal to $y_1[n] + y_2[n]$.

In general consider $x_1[n]$ and $x_2[n]$ as the input sequences and the corresponding outputs are $y_1[n]$ and $y_2[n]$ respectively. We know that $y_i[n] = ny_i[n-1] + x_i[n]$ for $i = \{1, 2\}$. Let us consider another input sequence $x_3[n] = ax_1[n] + bx_2[n]$ and let the corresponding output be $y_3[n]$. In order to find $y_3[n]$ we need to find a sequence which satisfies $y_3[n] - ny_3[n-1] - x_3[n] = 0$. Let us check if $ay_1[n] + by_2[n]$ satisfies this equation. We first substitute $ay_1[n] + by_2[n]$ in the LHS to get $ay_1[n] + by_2[n] - nay_1[n-1] - nby_2[n-1] - x_3[n]$. Substitute $x_3[n] = ax_1[n] + bx_2[n]$ to get $a(y_1[n] - ny_1[n-1] - x_1[n]) + b(y_2[n] - ny_2[n-1] - x_2[n])$. We know $y_i[n] = ny_i[n-1] + x_i[n]$ for $i = \{1, 2\}$. Hence, $a(y_1[n] - ny_1[n-1] - x_1[n]) + b(y_2[n] - ny_2[n-1] - x_2[n]) = 0$. Note that the output corresponding to any input sequence is unique because the system is relaxed.

(c) Is the system time-invariant? Justify your answer.

To determine if the system is time invariant, consider the input $\delta[n-1]$. The corresponding output $y_d[n]$ is computed as follows. $y_d[n] = 0$ for all $n < 0$. For $n > 0$

$$y_d[0] = 0,\ y_d[1] = 1,\ y_d[2] = 2,\ y_d[3] = 6,\ y_d[4] = 24..$$

We know from part i) that output to $\delta[n]$ is $y[n] = n!u[n]$. Observe that the above sequence $y_d[n] \neq y[n-1] = (n-1)!u[n-1]$

(d) Is the system BIBO stable? Justify your answer.

From part i) the output for $\delta[n]$ is $y[n] = n!u[n]$. Clearly the input is bounded, but there does not exist a bound $M$ on $y[n]$ because $y[n]$ for large enough $n$ will exceed any finite value $M$.