EE113: Digital Signal Processing Spring 2011

Solution to Final Exam (Version A)

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1. Problem 1

   (a) Since $y(n) = x^2(n)$, we have $Y(e^{j\omega}) = X(e^{j\omega}) * X(e^{j\omega})$. Hence, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does. To get $y_c(n) = x_c^2(n)$, the highest frequency of $y_c(n)$ is $0.5/T =$ 5000 Hz. Consequently, the highest frequency of $x_c(n)$ is 2500 Hz, or $\pi \times 5000$ rads/second.

   (b) The magnitude response is

   $$\left|H(e^{j\omega})\right| = \frac{\sqrt{2 - 2\cos(\omega - \omega_0)} \cdot \sqrt{2 - 2\cos(\omega + \omega_0)}}{\sqrt{1.81 - 1.8\cos(\omega - \omega_0)} \cdot \sqrt{1.81 - 1.8\cos(\omega + \omega_0)}}.$$

   The phase response is

   $$\begin{aligned}\angle H(e^{j\omega}) &= \tan^{-1}\left(\frac{\sin(\omega - \omega_0)}{1 - \cos(\omega - \omega_0)}\right) + \tan^{-1}\left(\frac{\sin(\omega + \omega_0)}{1 - \cos(\omega + \omega_0)}\right) \\ &- \tan^{-1}\left(\frac{0.9\sin(\omega - \omega_0)}{1 - 0.9\cos(\omega - \omega_0)}\right) - \tan^{-1}\left(\frac{0.9\sin(\omega + \omega_0)}{1 - 0.9\cos(\omega + \omega_0)}\right).\end{aligned}$$

   (c) Since $f = 60$ Hz in frequency domain corresponds to $\Omega = 2\pi f =$ $2\pi \times 60$ rads/second in angular frequency domain, we have

   $$\omega = \Omega T = 2\pi \times 60 \times 10^{-4} = \frac{3\pi}{250}$$

   in the transform domain of DTFT.

   To filter the 60Hz out, we need

   $$|H(e^{j\omega})|\big|_{\omega = \frac{3\pi}{250}} = 0,$$

   which holds true if

   $$\cos(\omega - \omega_0)|_{\omega = \frac{3\pi}{250}} = 1.$$

   Hence, we have $\omega_0 = \frac{3\pi}{250}$.

2. Problem 2

   (a) $H_1(e^{jw})$ corresponds to a frequency shifted version of $H_{lp}(e^{j\omega})$, specifically:

   $$H_1(e^{jw}) = H_{lp}(e^{j(w-\pi)})$$

   Thus, we have

   $$H_1(e^{jw}) = \begin{cases} 0, & |w| \leq 0.8\pi \\ 1, & 0.8\pi < |w| \leq \pi \end{cases}$$

   This is a highpass filter.

   (b) $H_2(e^{jw})$ corresponds to a frequency modulated version of $H_{lp}(e^{j\omega})$, specifically:

   $$H_2(e^{jw}) = H_{lp}(e^{jw}) * (\delta(w - 0.5\pi) + \delta(w + 0.5\pi)), \quad |w| \leq \pi$$

   Thus, we have

   $$H_2(e^{jw}) = \begin{cases} 0, & |w| \leq 0.3\pi \\ 1, & 0.3\pi < |w| < 0.7\pi \\ 0, & 0.7\pi \leq |w| \leq \pi \end{cases}$$

   This is a bandpass filter.

   (c) $H_3(e^{jw})$ corresponds to a periodic convolution of $H_{lp}(e^{j\omega})$ with another lowpass filter, specifically:

   $$H_3(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) H_{lp}(e^{j(w-\theta)}) \, d\theta$$

   where $H(e^{jw})$ is given by

   $$H(e^{jw}) = \begin{cases} 1, & |w| < 0.1\pi \\ 0, & 0.1\pi \leq |w| \leq \pi \end{cases}$$

   Carrying out the convolution, we get

   $$H_3(e^{jw}) = \begin{cases} 0.1, & |w| < 0.1\pi \\ -\frac{|w|}{2\pi} + 0.15, & 0.1\pi \leq |w| \leq 0.3\pi \\ 0, & 0.3\pi < |w| \leq \pi \end{cases}$$

   This is lowpass filter.

(d) The energy of $h_3(n)$ is computed using Parseval theorem

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\begin{aligned}
E =& \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_3(e^{jw})|^2 dw \\
=& \frac{1}{2\pi}\left(\int_{-0.1\pi}^{0.1\pi} (0.1)^2 + 2\Big(\int_{0.1\pi}^{0.3\pi} (0.15^2 - \frac{0.3}{2\pi}w + \frac{1}{4\pi^2}w^2)\,dw\Big)\right)
\end{aligned}
$$

The answer is considered to be complete up to the above equality. Further calculation gives us the energy of $h_3(n)$ as 0.005.

3. Problem 3. Let $h(n) = \delta(n) - \frac{1}{2}\delta(n - n_0)$

   (a) The Z-transform of $h(n)$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ &= 1 - \frac{1}{2} z^{-n_0} \end{aligned}$$

   The N-point DFT of $h(n)$ with $N = 4n_0$ is

$$\begin{aligned} H(k) &= \sum_{n=0}^{4n_0 - 1} h(n) e^{-j\frac{2\pi k n}{4n_0}}, \quad 0 \leq k \leq 4n_0 - 1 \\ &= 1 - \frac{1}{2} e^{-j(\pi/2)k} \end{aligned}$$

   (b) The impulse response of $h_i(n)$ is

$$\begin{aligned} H_i(z) &= \frac{1}{1 - 1/2 z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \\ h_i(n) &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta(n - k n_0) \end{aligned}$$

   This is IIR filter, so infinite duration.

   (c)

$$G(k) = \frac{1}{H(k)} = \frac{1}{1 - \frac{1}{2} e^{-j(\pi/2)k}}, \quad 0 \leq k \leq 4n_0 - 1$$

   The impulse response, $g(n)$ is just $h_i(n)$ time-aliased by $4n_0$ points

$$\begin{aligned} g(n) &= \left(1 + \frac{1}{16} + \frac{1}{256} + \ldots\right)\delta(n) + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \ldots\right)\delta(n - n_0) \\ &\quad + \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \ldots\right)\delta(n - 2n_0) + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \ldots\right)\delta(n - 3n_0) \\ &= \frac{16}{15}\delta(n) + \frac{8}{15}\delta(n - n_0) + \frac{4}{15}\delta(n - 2n_0) + \frac{2}{15}\delta(n - 3n_0) \end{aligned}$$

   (d)

$$\begin{aligned} y(n) = g(n) * h(n) &= \frac{16}{15}\delta(n) + \frac{8}{15}\delta(n - n_0) + \frac{4}{15}\delta(n - 2n_0) + \frac{2}{15}\delta(n - 3n_0) \\ &\quad - \frac{8}{15}\delta(n - n_0) - \frac{4}{15}\delta(n - 2n_0) - \frac{2}{15}\delta(n - 3n_0) - \frac{1}{15}\delta(n - 4n_0) \\ &= \frac{16}{15}\delta(n) - \frac{1}{15}\delta(n - 4n_0) \end{aligned}$$

Since $y(n) \neq \delta(n)$, we cannot perfectly recover $x(n)$ from $y(n)$ using $g(n)$.

Indeed,

$$G(k)H(k) = 1, \quad 0 \leq k \leq 4n_0 - 1$$

However, this relationship is only true at $4n_0$ distinct frequencies. This fact does not imply that for all frequencies w:

$$G(e^{jw})H(e^{jw}) = 1$$

4. Problem 4 Let $x(n) = 0,\ n < 0,\ n > 7$ be a real eight-point sequence and let $X(k)$ be its eight-point DFT

   (a)

$$\frac{1}{8}\sum_{k=0}^{7} X(k)e^{j(2\pi/8)k\times 9} = \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j(2\pi/8)k\times 1} = x(1)$$

   (b)

$$\begin{aligned} V(k) &= X(z)\Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=-\infty}^{n=\infty} x(n)z^{-n}\Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^{n=7} x(n)z^{-n}\Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^{n=7} x(n)(2e^{j\pi/8})^{-n}e^{-j\frac{2\pi k}{8}n} \\ &= \sum_{n=0}^{n=7} v(n)e^{-j\frac{2\pi k}{8}n} \end{aligned}$$

   Thus, we have

$$v(n) = x(n)(2e^{j\pi/8})^{-n}$$

   (c)

$$\begin{aligned} w(n) &= \frac{1}{4}\sum_{k=0}^{3} W(k)e^{j\frac{2\pi kn}{4}} \\ &= \frac{1}{4}\sum_{k=0}^{3}(X(k)+X(k+4))e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=0}^{3} X(k+4)e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=4}^{7} X(k)e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}\sum_{k=0}^{7} X(k)e^{j\frac{2\pi}{4}k2n} \\ &= 2x(2n) \end{aligned}$$

Thus,

$$w(n) = 2x(2n)$$

(d) Note that $Y(k)$ can be written as

$$\begin{aligned} Y(k) &= X(k) + (-1)^k X(k) \\ &= X(k) + e^{-j\frac{2\pi k4}{8}} X(k) \end{aligned}$$

Using DFT properties, we have

$$y(n) = x(n) + x((n-4) \bmod 8)$$