SOLUTIONS FOR MIDTERM EXAMINATION

1. A causal system is described by the following second-order difference equation:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n), \qquad y(-1) = 0, \quad y(-2) = 8, \quad n \geq 0$$

where $x(n)$ denotes causal input sequences.

(a) Is the system relaxed? Why or why not?
No, the system is not relaxed because it has nonzero initial conditions.

(b) Is the system linear? Why or why not?
No, the system is not linear because it has nonzero initial conditions.

(c) What are the modes of the system?
The characteristic equation of the system is given by

$$\lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = 0$$

Solving it gives the modes of the system:

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = \frac{1}{4}$$

(d) Determine the zero-input solution of the system. Call it $y_{zi}(n)$.
The general homogeneous solution of the system has the form:

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n, \qquad \forall n$$

Substituting the initial conditions, we get

$$\begin{cases} n = -1, & C_1\left(\frac{1}{2}\right)^{-1} + C_2\left(\frac{1}{4}\right)^{-1} = 0 \\ n = -2, & C_1\left(\frac{1}{2}\right)^{-2} + C_2\left(\frac{1}{4}\right)^{-2} = 8 \end{cases}$$

Solving the linear equations yields

$$C_1 = -2, \qquad C_2 = 1$$

Therefore, the zero-input solution of the system is given by

$$y_{zi}(n) = \left[-2\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right]u(n)$$

(e) Determine the zero-state solution of the system when $x(n) = \left(\frac{1}{3}\right)^n u(n)$. Call it $y_{zs}(n)$. Since the zero-state solution is the output of an LTI system, it can be calculated by

$$y_{zs}(n) = x(n) \star h(n)$$

where $h(n)$ is the impulse response of the zero-state system. By definition, it holds that

$$h(n) - \frac{3}{4}h(n-1) + \frac{1}{8}h(n-2) = \delta(n), \qquad \text{relaxed}$$

Since the system is relaxed and the input is $\delta(n)$, we get

$$h(n) = 0, \quad n < 0, \qquad \text{and} \qquad h(0) = 1$$

When $n > 0$, the difference equation becomes

$$h(n) - \frac{3}{4}h(n-1) + \frac{1}{8}h(n-2) = 0, \qquad n > 0$$

which is identical to the homogeneous equation we solved in part (d). Therefore, the general homogeneous solution $y_h(n)$ from part (d) also applies here. Substituting the new initial conditions, i.e.,

$$h(-1) = 0, \qquad h(0) = 1$$

into $y_h(n)$, we get

$$\begin{cases} n = -1, & C_1 \left(\frac{1}{2}\right)^{-1} + C_2 \left(\frac{1}{4}\right)^{-1} = 0 \\ n = 0, & C_1 + C_2 = 1 \end{cases}$$

Solving the linear equations yields

$$C_1 = 2, \qquad C_2 = -1$$

Therefore, the impulse response is given by

$$h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n)$$

The zero-state solution for $x(n) = \left(\frac{1}{3}\right)^n u(n)$ is then given by

$$\begin{aligned} y_{zs}(n) &= x(n) \star h(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \left\{\sum_{k=0}^{n} \left[2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right] \left(\frac{1}{3}\right)^{n-k}\right\} u(n) \\ &= \left(\frac{1}{3}\right)^n \left[2\sum_{k=0}^{n}\left(\frac{3}{2}\right)^k - \sum_{k=0}^{n}\left(\frac{3}{4}\right)^k\right] u(n) \\ &= \left(\frac{1}{3}\right)^n \left[2\frac{\left(\frac{3}{2}\right)^{n+1} - 1}{\left(\frac{3}{2}\right) - 1} - \frac{\left(\frac{3}{4}\right)^{n+1} - 1}{\left(\frac{3}{4}\right) - 1}\right] u(n) \\ &= \left[6\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{4}\right)^n\right] u(n) \end{aligned}$$

(f) Determine the complete solution of the system to the input sequence of part (e). When $n \geq 0$, the complete solution is given by

$$\begin{aligned} y(n) &= y_{zi}(n) + y_{zs}(n) \\ &= \left[-2\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + 6\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{4}\right)^n \right] u(n) \\ &= \left[4\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n + 4\left(\frac{1}{4}\right)^n \right] u(n) \end{aligned}$$

(g) Determine the z-transform and ROC of the complete response determined in part (f). The z-transform of $y(n)$ from part (f) is given by

$$Y(z) = \frac{4z}{z - \frac{1}{2}} - \frac{8z}{z - \frac{1}{3}} + \frac{4z}{z - \frac{1}{4}}$$

where the ROC is given by

$$|z| > \frac{1}{2}$$

(h) Determine the value of $y_{zs}(0)$ in at least three different ways.

i. Using the difference equation. From the difference equation that is given in part (e), the $y_{zs}(0)$ satisfies the following equality:

$$y_{zs}(0) - \frac{3}{4}y_{zs}(-1) + \frac{1}{8}y_{zs}(-2) = x(0)$$

with zero initial conditions:

$$y_{zs}(-1) = 0, \qquad y_{zs}(-2) = 0$$

and input:

$$x(0) = 1$$

It is straightforward to determine the value of $y_{zs}(0)$ by

$$y_{zs}(0) = 1$$

ii. Using the closed-form expression of $y_{zs}(n)$. From part (e), we get

$$y_{zs}(0) = 6 - 8 + 3 = 1$$

iii. Using the z-transform of $y_{zs}(n)$. The z-transform of $y_{zs}(n)$ is given by

$$Y_{zs}(z) = \frac{6z}{z - \frac{1}{2}} - \frac{8z}{z - \frac{1}{3}} + \frac{3z}{z - \frac{1}{4}}$$

where the ROC is given by

$$|z| > \frac{1}{2}$$

Using the initial value theorem, we get

$$y_{zs}(0) = \lim_{z \to \infty} Y_{zs}(z) = \lim_{z \to \infty} \left[\frac{6z}{z - \frac{1}{2}} - \frac{8z}{z - \frac{1}{3}} + \frac{3z}{z - \frac{1}{4}} \right] = 6 - 8 + 3 = 1$$

where we used the l'Hospital's rule.

(i) Without redoing any calculation, what would the complete response be if the initial conditions are each multiplied by a factor of 2 and the input sequence of part (e) is multiplied by a factor of 4.

By the superposition property of the zero-input solution, it will be scaled by the same factor of 2:

$$y'_{zi}(n) = 2y_{zi}(n)$$

Likewise, by the superposition property of the zero-state solution, it will be scaled by a factor of 4:

$$y'_{zs}(n) = 4y_{zs}(n)$$

Therefore, the new complete response is given by

$$\begin{aligned} y'(n) &= y'_{zi}(n) + y'_{zs}(n) \\ &= 2y_{zi}(n) + 4y_{zs}(n) \\ &= \left[-4\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{4}\right)^n + 24\left(\frac{1}{2}\right)^n - 32\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n\right] u(n) \\ &= \left[20\left(\frac{1}{2}\right)^n - 32\left(\frac{1}{3}\right)^n + 14\left(\frac{1}{4}\right)^n\right] u(n) \end{aligned}$$

2. True or False? Explain or give counter-examples:

(a) Every causal system is relaxed.

False. Consider $y(n) - \frac{1}{2}y(n-1) = x(n)$, $y(-1) = 1$, $n \geq 0$. The system is causal, but not relaxed since the initial conditions are non-zero.

(b) Every relaxed system is causal.

False. Consider the system $y(n) = x(n) \cdot x(n+1)$. This system is relaxed since the output is zero as long as the input is zero. On the other hand, the system is non-causal.

(c) LTI systems that are causal are also relaxed.

True. LTI systems are fully described by their impulse-response sequences $h(n)$. When the system is causal, then $h(n)$ is a causal sequence. Now, the output of the system to an arbitrary input $x(n)$ is given by:

$$\begin{aligned} y(n) &= h(n) \star x(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=0}^{\infty} h(k)x(n-k) \qquad \text{[Since the system is causal]} \\ &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \end{aligned}$$

which only depends on current and past values of $x(n)$. This means that the system's output $y(n)$ will be zero unless $x(k) \neq 0$ for some $k \leq n$.

(d) An LTI system is BIBO stable if, and only if, all its poles lie inside the unit circle.

False. Consider the non-causal system $h(n) = \delta(n+1)$, which has a pole at infinity. Clearly, the system is stable since

$$\sum_{k=-\infty}^{\infty} |h(k)| = 1 < \infty$$

but the system has a pole outside the unit-circle.

(e) If the response to the zero sequence is the zero sequence, then the system is relaxed.

False. This is a necessary condition for the system to be relaxed, but it is not sufficient. For example, consider $y(n) = x(n+1)$.

(f) If the response to the zero sequence is the zero sequence, the system can still be nonlinear.

True. Consider $y(n) = [x(n)]^2$.

(g) Every memoryless system is necessarily causal.

True. Since the system only depends on the input via $x(n)$, the system is causal.

(h) Every memoryless system is necessarily relaxed.

False. Consider $y(n) = 1 + x(n)$. System is memoryless, but not relaxed.

3. Let $x(n)$ denote the height (in some convenient units) of the n-th student entering the classroom at the beginning of the lecture. Let $y(n)$ be the average height of the first n students who entered the room.

(a) Find a difference equation relating $y(n)$ to $y(n-1)$ and $x(n)$ for $n \geq 1$.

We have that $y(n)$ is the average height of the first n student; i.e,

$$y(n) = \frac{1}{n}\sum_{k=1}^{n} x(k) = \frac{1}{n}\sum_{k=1}^{n-1} x(k) + \frac{1}{n}x(n)$$

We know that $y(n-1)$ satisfies

$$y(n-1) = \frac{1}{n-1}\sum_{k=1}^{n-1} x(k)$$

Using the above form of $y(n-1)$, we have that $y(n)$ satisfies:

$$y(n) = \frac{n-1}{n}y(n-1) + \frac{1}{n}x(n), \quad y(0) = 0, n \geq 1$$

(b) Is the system causal? time-invariant? linear? relaxed?

The system is causal since it only requires knowledge of previous inputs.

The system is time-varying. Consider the original output

$$y(n) = \frac{1}{n}\sum_{k=1}^{n} x(k)$$

Now, the shifted version of the output $y_m(n) = y(n+m)$ is given by

$$y_m(n) = \frac{1}{n+m}\sum_{k=1}^{n+m} x(k)$$

On the other hand, the output of the system to the shifted input $x(n+m)$ is:

$$y'(n) = \frac{1}{n}\sum_{k=1}^{n} x(k+m) = \frac{1}{n}\sum_{k=m+1}^{m+n} x(k)$$

Clearly, $y'(n) \neq y_m(n)$ in general.
The system is linear. Consider the original form of the system:

$$y(n) = \frac{1}{n}\sum_{k=1}^{n} x(k)$$

If a linear combination of inputs $ax_1(n) + bx_2(n)$ is applied to the system, the output is a linear combination of the corresponding outputs:

$$y(n) = \frac{1}{n}\sum_{k=1}^{n}[ax_1(k) + bx_2(n)] = \frac{a}{n}\sum_{k=1}^{n} x_1(k) + \frac{b}{n}\sum_{k=1}^{n} x_2(n) = ay_1(n) + by_2(n)$$

where $x_1(n) \to y_1(n)$ and $x_2(n) \to y_2(n)$.
The system is relaxed since it is causal and its initial conditions are zero.