66 average

Total: 100 points

ECE113: Digital Signal Processing

Midterm 1

10:00 am - 11:40 am, Feb 04, 2019

NAME: ______________________ **UID:** ______________

This exam has 3 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points	
1	32 +8	40	40
2	14+ 8	35	22
3	10	25	10
Total	72	100	

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (40 points): There are three sub-problems to this problem, which are not related.

(a) **(12 points)** If the signal $x[n]$ is real and odd, are the following signals even or odd? (Justify why)

$x[-n] = -x[n]$

(i) $y_1[n] = x[n^2]$

(ii) $y_2[n] = x[3n]$

(iii) $y_3[n] = \begin{cases} x[-n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$

12

(i) $y_1[n] = x[n^2]$ ✓

$y_1[-n] = x[(-n)^2] = x[n^2] = y_1[n]$

even

(ii) $y_2[n] = x[3n]$ ✓

$y_2[-n] = x[-3n] = -x[3n] = -y_2[n]$

odd

(iii) $y_3[n] = \begin{cases} x[-n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$

$y_3[-n] = \begin{cases} x[-\frac{(-n)}{2}], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$

$y_3[-n] = \begin{cases} x[n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$ ✓

$y_3[-n] = -y_3[n]$

since $y_3[n] = \begin{cases} -x[n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$

odd

20

(b) **(20 points)** Determine whether the system

$$y[n] = |x[n]|$$

where $|x[n]|$ is the magnitude of the input $x[n]$, is

(i) Linear

(ii) Causal

(iii) Stable

(iv) Time invariant

(i) $Sys\{ax_1[n]\} = |ax_1[n]|$ $Sys\{bx_2[n]\} = |bx_2[n]|$

$Sys\{ax_1[n] + bx_2[n]\} = |ax_1[n] + bx_2[n]| \stackrel{?}{=} |ax_1[n]| + |bx_2[n]|$ ✓

Assume $a=1$, $b=1$, $x_1[n] = -2$, $x_2[n] = 2$

$|ax_1[n] + bx_2[n]| = 0$ $|ax_1[n]| + |bx_2[n]| = 4$

not linear

(ii) It does not depend on future values of n, so Causal ✓

(iii) Assume $||x[n]|| \leq B_x < \infty$ for all n

$|y[n]| = ||x[n]||$

$|y[n]| = \underbrace{B_x}_{B_y} < \infty$ Stable ✓

(iv) $Sys\{x[n-d]\} = |x[n-d]|$

$y[n-d] = |x[n-d]|$

$Sys\{x[n-d]\} = y[n-d]$ ✓

Time invariant

(c) **(8 points)** Assume that we are given the values of $x[2n]$ and $x[n^2]$ for all n: can we reconstruct the signal $x[n]$ using these values?

$X[2n]$ is downsampling, so we lose the odd n values of $x[n]$

For $x[n^2]$,

$n = 0 \quad x[0]$
$n = 1 \quad x[1]$
$n = 2 \quad x[4]$
$n = 3 \quad x[9]$
$n = 4 \quad x[16]$
$\vdots$
$n = a \quad x[a^2]$

As you can see, we lose values such as $x[3]$, $x[5]$, $x[7]$, ... We can't get these values from $x[2n]$ either, so we cannot reconstruct the signal $x[n]$ using these values.

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Problem 2 (35 points) : The following two questions are not related.

(a) **(20 points)** The response of an LTI system to the input $x_1[n] = u[n]$ is $y_1[n] = a^n u[n]$ where $a < 1$. What is the response of the system when the input is $x_2[n] = b^n u[n]$ where $b < 1$?

You may find the formula for finite geometric sum useful: $\sum_{j=p}^{q} \beta^j = \frac{\beta^p - \beta^{q+1}}{1-\beta}$.

$Sys\{u[n]\} = a^n u[n]$

$Sys\{b^n u[n]\} = ?$ $\qquad h[n] = a^n \delta[n]$

$x_1[n] * h[n] = y_1[n]$

$h[n] = ?$

$h[n] = Sys\{\delta[n]\} = Sys\{u[n] - u[n-1]\}$

$h[n] = a^n u[n] - a^{n-1} u[n-1]$ $\qquad$ Let $h_1[n] = a^n u[n]$ $\quad h_2[n] = -a^{n-1} u[n-1]$

$Sys\{b^n u[n]\} = x_2[n] * h[n] = x_2[n] * (h_1[n] + h_2[n])$

$Sys\{b^n u[n]\} = x_2[n] * h_1[n] + x_2[n]\, h_2[n]$

$Sys\{b^n u[n]\} = \underbrace{\sum_{k=-\infty}^{+\infty} x_2[k] h_1[n-k]}_{z_1[n]} + \underbrace{\sum_{k=-\infty}^{+\infty} x_2[k] h_2[n-k]}_{z_2[n]}$

$h_1[-k]$

$x_2[k]$

$n=0$, 0

$n=1$, $-a^0 b^0$

$n=2$, $-a^0 b^1$

$z_2[n] = -z[n-1] = -\sum_{k=0}^{n-1} a^k b^{n-1-k}$

$n=0$, $a^0 b^0$

$n=1$, $a^1 b^0 + a^0 b^1$

$n=2$, $a^2 b^0 + a^1 b^1 + a^0 b^2$

$y_2[n] = Sys\{b^n u[n]\} = \left(\sum_{k=0}^{n} a^k b^{n-k}\right) - \left(\sum_{k=0}^{n-1} a^k b^{n-1-k}\right)$

$z_1[n] = \sum_{k=0}^{n} a^k b^{n-k}$

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(b) **(15 points)** Consider two periodic signals $x[n]$ and $y[n]$, both with period N. Define the signal $z[n]$ as the periodic convolution of $x[n]$ and $y[n]$, i.e.,

$$z[n] = x[n] \circledast y[n]$$

Given the signal $z[n]$ and the DTFS spectrum of $y[n]$, is it always possible to uniquely determine the signal $x[n]$? Explain why/why not?

$$z[n] = \sum_{k=0}^{N-1} x[k]\, y[n-k]$$

$$z[0] = \sum_{k=0}^{N-1} x[k]\, y[-k] = \left(x[0]y[0] + x[1]y[-1] + \ldots + x[N-1]\, y[-(N-1)]\right)$$

$$z[1] = \sum_{k=0}^{N-1} x[k]\, y[1-k] = \left(x[0]y[1] + x[1]y[0] + \ldots x[N-1]\, y[1-(N-1)]\right)$$

We would need to find $x[0], x[1], \ldots x[N-1]$, we know $z[n]$ and can reconstruct $y[n]$, As you can see, we can keep writing out equations as shown above until we have enough equations for $N-1$ $x[n]$ terms then solve for each of them. We can then reconstruct $x[n]$ since it is periodic with period N. ✗

↓

not necessarily.

N equ, N variables need not have unique solution

(eg) $x+y=0$

$2x+2y=0$

(8)

Problem 3 (25 points) : Let us say there is a *real* periodic signal $x[n]$, with period $N = 7$. You do not know the signal and wish to determine it. However, you happen to know that the average value of the signal over each period is 0, i.e.,

$$\frac{1}{7}\sum_{n=0}^{6} x[n] = 0$$

Assume that the DTFS coefficients of $x[n]$ are $c_k,\ k \in \{0, 1, ..., 6\}$. Of course, you do NOT know these coefficients. As you are trying to determine the signal, your friend tells you that he/she tried to reconstruct $x[n]$ using its first four DTFS coefficients, i.e., your friend's reconstruction is

$$\tilde{x}[n] = \sum_{k=0}^{3} c_k e^{j\frac{2\pi}{7}kn}$$

Your friend agreed to give you $\tilde{x}[n]$, but not any of the coefficients c_k. After thinking for a few minutes, you realize that $x[n]$ can be obtained using your friend's reconstruction $\tilde{x}[n]$. Explain and derive the expression of $x[n]$ in terms of $\tilde{x}[n]$?

$\tilde{x}[0] = \sum_{k=0}^{3} c_k = c_0 + c_1 + c_2 + c_3$ ✓

$\tilde{x}[1] = \sum_{k=0}^{3} c_k e^{j\frac{2\pi}{7}k} = c_0 + c_1 e^{j\frac{2\pi}{7}} + c_2 e^{j4\pi/7} + c_3 e^{j6\pi/7}$

$\tilde{x}[2] = \sum_{k=0}^{3} c_k e^{j4\pi/7 k} = c_0 + c_1 e^{4\pi/7 j} + c_2 e^{8\pi/7 j} + c_3 e^{j\frac{12\pi}{7}}$

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$\tilde{x}[3] = \sum_{k=0}^{3} c_k e^{j6\pi/7 k} = c_0 + c_1 e^{j6\pi/7} + c_2 e^{j12\pi/7} + c_3 e^{j18\pi/7}$

$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \frac{1}{7}\sum_{n=0}^{6} x[n] e^{-j\frac{2\pi k}{7}}$

$c_0 = \frac{1}{7}\sum_{n=0}^{6} x[n] = 0$ ✓

$c_0 = 0$

not necessary.

Since $\frac{1}{7}\sum_{n=0}^{6} x[n] = 0$, we know the signal is odd, so we know

c_1 and c_6 cancel

c_2 and c_5 cancel

c_3 and c_4 cancel

From the summations, we can solve for c_1, c_2, c_3 ✓ then ~~c_6, c_5, c_4~~ Then we find $x[n]$ with the formula $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$. We know it is period 7, we find $x[0] \ldots x[6]$.

how? signal is not odd.

Total: 100 points

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Problem	Points	Total Points
1		40
2		35
3		25
Total		100

Extra Pages: _________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (40 points): There are three sub-problems to this problem, which are not related.

(a) **(12 points)** If the signal $x[n]$ is real and odd, are the following signals even or odd? (Justify why)

(i) $y_1[n] = x[n^2]$

(ii) $y_2[n] = x[3n]$

(iii) $y_3[n] = \begin{cases} x[-n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$

Solutions : $x[n]$ is real and odd. $x[-n] = -x[n]$.

i. Let $y_1[n] = x[n^2]$.

$$y_1[-n] = x[(-n)^2] = x[n^2] = y_1[n]$$

$y_1[n] = x[n^2]$ is even.

ii. Let $y_2[n] = x[3n]$.

$$y_2[-n] = x[3(-n)] = x[-3n] = -x[3n] = -y_2[n]$$

$y_2[n] = x[3n]$ is odd.

iii. Let $y_3[n] = x[-n/2]$.

$$y_3[-n] = x[-(-n)/2] = -x[-n/2] = -y_3[n], \quad \text{if } n/2 \text{ is an integer}$$

$y_3[n] = x[-n/2]$ is odd.

Common Mistakes :

- We need x[0]=0 for a signal to be odd, but that is not a sufficient condition.
- Instead of proving it is even or odd, if we take an example, and because the example worked out, it cannot be said, in general, that a signal is even or odd.

(b) **(20 points)** Determine whether the system

$$y[n] = |x[n]|$$

where $|x[n]|$ is the magnitude of the input $x[n]$, is

(i) Linear

(ii) Causal

(iii) Stable

(iv) Time invariant

Solutions :

$$y[n] = |x[n]|$$

(i)

$$\begin{aligned} y_1[n] &= Sys\{x_1[n]\} = |x_1[n]| \\ y_2[n] &= Sys\{x_2[n]\} = |x_2[n]| \\ \text{Let } x_3[n] &= ax_1[n] + bx_2[n] \\ y_3[n] = Sys\{x_3[n]\} &= |x_3[n]| \\ &= |ax_1[n] + bx_2[n]| \\ &\neq a|x_1[n]| + b|x_2[n]| = ay_1[n] + by_2[n] \end{aligned}$$

The system is not linear.

(ii) Since the system depends only on the present input, it is causal.

(iii) Assume $|x[n]| \leq B_x < \infty$, then

$$|y[n]| = |x[n]| \leq B_x < \infty$$

Hence the system is stable.

(iv)

$$\begin{aligned} y[n] &= Sys\{x[n]\} = |x[n]| \\ y_K[n] &= Sys\{x_K[n]\} = Sys\{x[n-K]\} = |x[n-K]| = y[n-K] \end{aligned}$$

The system is time-invariant.

Common Mistakes :

- System is "sometimes linear" is an incorrect statement.
- Proving linearity assuming positive coefficients and positive signal is incorrect
- Proving stability and causality by finding the impulse response, and applying impulse response properties - this we cannot do for systems that are NOT LTI

(c) **(8 points)** Assume that we are given the values of $x[2n]$ and $x[n^2]$ for all n: can we reconstruct the signal $x[n]$ using these values?

Solutions :No, we cannot reconstruct the signal $x[n]$, because we cannot obtain $x[-1]$, $x[3]$, $x[5]$, and so on $\ldots$, from the given signals $x[2n]$ and $x[n^2]$

Common Mistakes :

- Time scaling and time shifting $x[2n]$ and $x[n^2]$ will not provide the missing samples.
- Although, $x[2n]$ is odd signal, and $x[n^2]$ is even signal, $x[n] \neq x[2n] + x[n^2]$, i.e., a sum of even and odd signal. Also, note that $x[n]$ is not given to be an odd signal for this part of problem 1.

Problem 2 (35 points) : The following two questions are not related.

(a) **(20 points)** The response of an LTI system to the input $x_1[n] = u[n]$ is $y_1[n] = a^n u[n]$ where $a < 1$. What is the response of the system when the input is $x_2[n] = b^n u[n]$ where $b < 1$?

You may find the formula for finite geometric sum useful: $\sum_{j=p}^{q} \beta^j = \frac{\beta^p - \beta^{q+1}}{1-\beta}$.

Solutions :We know that $\delta = u[n] - u[n-1]$, thus $\delta[n] = x_1[n] - x_1[n-1]$. The impulse response of the system hence is:

$$h[n] = y_1[n] - y_1[n-1] = a^n u[n] - a^{n-1} u[n-1]$$

With this, the response to input $x_2[n]$ is

$$y_2[n] = x_2[n] * h[n] = x_2[n] * y_1[n] - x_2[n] * y_1[n-1]$$

Now,

$$\begin{aligned} x_2[n] * y_1[n] &= b^n u[n] * a^n u[n] \\ &= \sum_{k=-\infty}^{\infty} b^k u[k] a^{n-k} u[n-k] \\ &= u[n] \sum_{k=0}^{n} b^k a^{n-k}, \end{aligned}$$

where the factor $u[n]$ is there because if $n < 0$, then $u[n-k] \cdot u[k] = 0$ for all k. Using the formula for finite geometric sum:

$$x_2[n] * y_1[n] = u[n] a^n \frac{1 - (b/a)^{n+1}}{1 - (b/a)} = u[n] \frac{a^{n+1} - b^{n+1}}{a - b}$$

From the properties of convolution $x_2[n] * y_1[n-1]$ is just going to be a right shifted version of $x_2[n] * y_1[n]$, i.e.,

$$x_2[n] * y_1[n-1] = u[n-1] \frac{a^n - b^n}{a - b}$$

Thus,

$$y_2[n] = \frac{a^{n+1} - b^{n+1}}{a - b} u[n] - \frac{a^n - b^n}{a - b} u[n-1]$$

Common Mistakes :

- In order to determine the impulse response

  $h[n] = a^n u[n] - a^{n-1} u[n-1] = a^n \delta[n] + a^n u[n-1] - a^{n-1} u[n-1] \neq \delta[n] - a^{n-1} u[n-1]$

- $u[n] \leftrightarrow a^n u[n]$

  It does not mean that because the system is LTI, $b^n u[n] \leftrightarrow b^n a^n u[n]$

- Look for the conditions on n in order to determine the response of the system $y_2[n]$ for all n.

(b) **(15 points)** Consider two periodic signals $x[n]$ and $y[n]$, both with period N. Define the signal $z[n]$ as the periodic convolution of $x[n]$ and $y[n]$, i.e.,

$$z[n] = x[n] \circledast y[n]$$

Given the signal $z[n]$ and the DTFS spectrum of $y[n]$, is it always possible to uniquely determine the signal $x[n]$? Explain why/why not?

Solutions :It is NOT uniquely possible to determine $x[n]$. In fact, we will show that it is not uniquely possible to determine the DTFS coefficients of $x[n]$, which in turn implies the statement.

Let the DTFS coefficients of $z[n]$ be e_k, the DTFS coefficients of $y[n]$ be d_k and the DTFS coefficients of $x[n]$ be c_k. We are given $z[n]$ and d_k, which means we know e_k and d_k. Now, by the periodic convolution property of DTFS, we know that

$$e_k = N c_k d_k$$

Given e_k and d_k, we can uniquely determine c_k if and only if $d_k \neq 0$. If $d_k = 0$, then for any c_k, $e_k = 0$. Hence, c_k can not be uniquely determined in this case.

Common Mistakes :

- Forgot the case when $d_k = 0$, in which case c_k can take any value.
- N equations and N variables need not have unique solution.

Problem 3 (25 points) : Let us say there is a *real* periodic signal $x[n]$, with period $N = 7$. You do not know the signal and wish to determine it. However, you happen to know that the average value of the signal over each period is 0, i.e.,

$$\frac{1}{7}\sum_{n=0}^{6} x[n] = 0$$

Assume that the DTFS coefficients of $x[n]$ are $c_k,\ k \in \{0, 1, ..., 6\}$. Of course, you do NOT know these coefficients. As you are trying to determine the signal, your friend tells you that he/she tried to reconstruct $x[n]$ using its first four DTFS coefficients, i.e., your friend's reconstruction is

$$\tilde{x}[n] = \sum_{k=0}^{3} c_k e^{j\frac{2\pi}{7}kn}$$

Your friend agreed to give you $\tilde{x}[n]$, but not any of the coefficients c_k. After thinking for a few minutes, you realize that $x[n]$ can be obtained using your friend's reconstruction $\tilde{x}[n]$. Explain and derive the expression of $x[n]$ in terms of $\tilde{x}[n]$?

Solutions :From the synthesis equation

$$x[n] = \sum_{k=0}^{6} c_k e^{j\frac{2\pi}{7}kn}$$

First note that $c_0 = \frac{1}{7}\sum_{n=0}^{6} x[n] = 0$ Thus,

$$\tilde{x}[n] = \sum_{k=1}^{3} c_k e^{j\frac{2\pi}{7}kn},$$

and

$$\begin{aligned} x[n] &= \sum_{k=1}^{6} c_k e^{j\frac{2\pi}{7}kn} \\ &= \sum_{k=1}^{3} c_k e^{j\frac{2\pi}{7}kn} + \sum_{k=4}^{6} c_k e^{j\frac{2\pi}{7}kn} \\ &= \tilde{x}[n] + \sum_{k=4}^{6} c_k e^{j\frac{2\pi}{7}kn}. \end{aligned}$$

Also, since $x[n]$ is real valued $c_k^* = c_{N-k} = c_{7-k}$. So the summation term in the equation above can be written as

$$\sum_{k=4}^{6} c_k e^{j\frac{2\pi}{7}kn} = \sum_{l=1}^{3} c_{7-l} e^{j\frac{2\pi}{7}(7-l)n} = \sum_{l=1}^{3} c_l^* e^{-j\frac{2\pi}{7}ln},$$

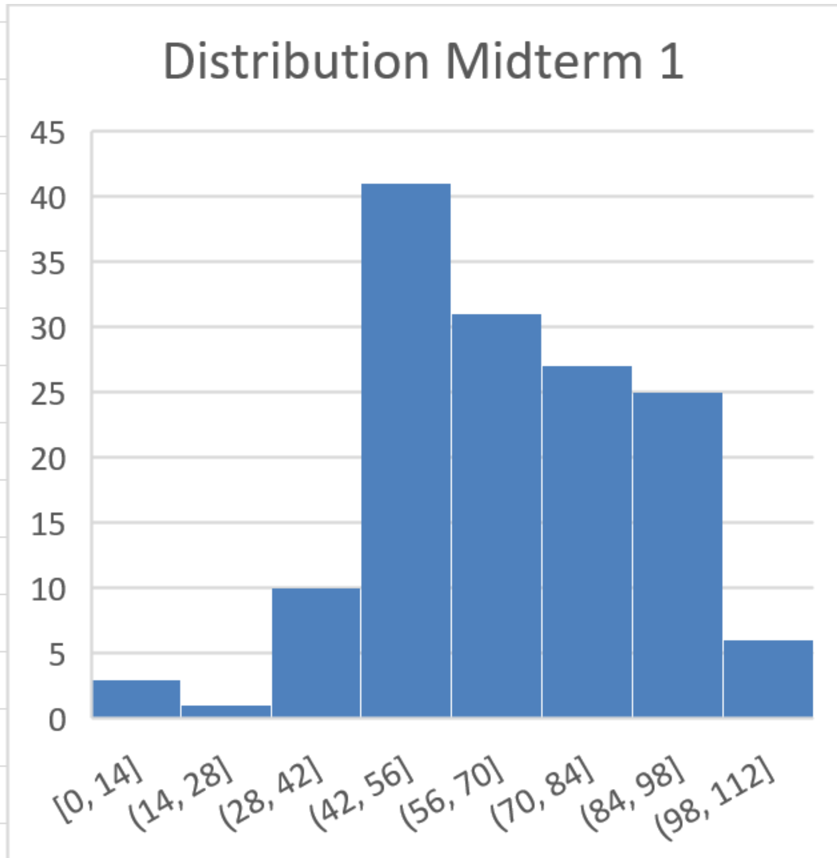
where in the first step we substituted $l = 7 - k$ and in the second step we used the conjugate symmetry property of c_k. Now note that $c_l^* e^{-j\frac{2\pi}{7}ln}$ is equal to $(c_l e^{j\frac{2\pi}{7}ln})^*$, and thus

$$\sum_{l=1}^{3} c_l^* e^{-j\frac{2\pi}{7}ln} = \left(\sum_{l=1}^{3} c_k e^{j\frac{2\pi}{7}ln}\right)^* = (\tilde{x}[n])^*$$

Thus, $x[n] = \tilde{x}[n] + (\tilde{x}[n])^*$.

Common Mistakes :

- Average value $= 0$ does not mean signal is odd.
- $c_0, \ldots, c_3$ does not reconstruct the first half of the signal.
- You can use analysis equation to reconstruct only $c_0, \ldots, c_3$ from $\tilde{x}[n]$.


Distribution Midterm 1
45
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