EE113: Digital Signal Processing Fall 2016
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MIDTERM EXAMINATION

1. (30 PTS) **True** or **False**? Explain or give counter-examples:

   (a) If $x(2n)$ is an energy sequence, then $x(n)$ is also an energy sequence.
   False. A suitable counter-example is:

   $$x(n) = \begin{cases} (\frac{1}{2})^n u(n) & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

   (b) If $x(n)$ is a periodic sequence then $x(2n+5)$ is also periodic.
   True. Assume $x(n)$ is periodic with period N. To prove that $x(2n+5)$ is periodic, we have to find α such that:

   $$x\left(2(n+\alpha)+5\right) = x\left(2n+2\alpha+5\right) = x\left(2n+5\right)$$

   for all n. Thus, $\alpha = \frac{N}{2}$ if N is even, otherwise, $\alpha = N$. Therefore, $x(2n+5)$ is periodic with period of at most α.

   (c) Every causal system is relaxed.
   False. A suitable counter-example is:

   $$y(n) = x(n) + 1$$

   which is causal but not relaxed.

   (d) Every time-invariant system is causal.
   False. A suitable counter-example is:

   $$y(n) = x(n+1)$$

   which is time-invariant, but not causal.

   (e) The series cascade of two time-variant linear systems can be LTI.
   True. An example of such a cascade is $S[\cdot] \triangleq S_2\left[S_1\left[\cdot\right]\right]$, where:

   $$S_1[x(n)] = x(n) + n$$
   $$S_2[x(n)] = x(n) - n$$

   Both $S_1[\cdot]$ and $S_2[\cdot]$ are time-variant, while $S[x(n)] = (x(n)+n) - n = x(n)$ is time-invariant.

   (f) The system $y(n) = y(n-1) + x^2(2n),\ y(-1) = 0,\ n \geq 0$, is time-invariant.
   False. Iterating from $n = 0$ onwards and incorporating the initial condition, we can write:

   $$y(n) = \sum_{k=0}^{n} x^2(2k), \quad n \geq 0$$

Then for $n \geq 0$

$$
\begin{aligned}
y_K(n) &= \sum_{k=0}^{n} x^2(2k-K) \\
&= \sum_{k=\lceil \frac{K}{2} \rceil}^{n} x^2(2k-K) \\
&= \sum_{k'=0}^{n-\lceil \frac{K}{2} \rceil} x^2\left(2\left(k' + \left\lceil \frac{K}{2} \right\rceil\right) - K\right) \\
&\neq \sum_{k'=0}^{n-K} x^2(2k') \\
&= y(n-K)
\end{aligned}
$$

so the system is time-variant.

Alternatively, we can provide a counter example: If $x(n) = \delta(n)$, then $y(n) = u(n)$. If $x_1(n) = x(n-1) = \delta(n-1)$ then $x_1(2n) = \delta(2n-1) = 0$ and $y_1(n) = 0 \neq y(n-1)$.

(g) $\{z : \frac{1}{2} < |z| < 2\}$ is the ROC of an anti-causal stable LTI system.
False. While the system is stable (ROC includes the unit circle), it is not anti-causal. The ROC of an anti-causal system has the form $\{z : |z| < \alpha\}$ for some constant α.

(h) The zero-state response of a system can be described using a convolution operation.
False. This only holds when the system is described by a constant-coefficient difference equation.

(i) Doubling the sampling period of a signal doubles the number of samples.
False. Doubling the sampling period results in less frequent sampling, hence a smaller number of samples (half the number of samples to be exact).

(j) The same constant-coefficient difference equation with boundary conditions can describe at most two systems.
True. The ambiguity left in the system after specifying the initial conditions is in the direction of time — it can run forward or backward.

2. (30 PTS) The following information is known about the behavior of a causal LTI system: (a) it has two modes at $\lambda_1 = 1/2$ and $\lambda_2 = 1/4$; (b) its response to $x(n) = (1/3)^n u(n)$ is an exponential sequence of the form $y(n) = \alpha^n u(n)$ with the largest possible energy value.

(a) Determine the value of α.
Solution. The z-transform of the impulse response has the generic form:

$$H(z) = \frac{N(z)}{D(z)} \tag{1}$$

The modes of the system determine:

$$D(z) = \left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right) \tag{2}$$

The z-transform of the output $Y(z)$ is then given by:

$$H(z)X(z) = \frac{N(z)}{(z-1/2)(z-1/4)} \cdot \frac{z}{z-1/3} = \frac{z}{z-\alpha} = Y(z) \tag{3}$$

We are free to chose $N(z)$ to ensure that the Eq. (3) is satisfied with α that maximizes the energy of $y(n)$. Since

$$E_y = \sum_{n=-\infty}^{\infty} \|y(n)\|^2 = \sum_{n=0}^{\infty} (\alpha^2)^n = \frac{1}{1-\alpha^2} \tag{4}$$

we are looking to maximize α. By looking at the poles in Eq. (3), we see that the only possibilities for α are 1/2, 1/3 and 1/4. Choosing $N(z) = (z-1/3)(z-1/4)$ allows us to cancel the poles at 1/3 and 1/4 so that we are left with:

$$Y(z) = \frac{(z-1/3)(z-1/4)}{(z-1/2)(z-1/4)} \cdot \frac{z}{z-1/3} = \frac{z}{z-1/2} \tag{5}$$

and in the time-domain:

$$y(n) = \left(\frac{1}{2}\right)^n u(n) \tag{6}$$

(b) Determine a constant coefficient difference equation for the system.
Solution. In part (a) we found that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z-1/3)(z-1/4)}{(z-1/2)(z-1/4)} = \frac{(1-1/3 \cdot z^{-1})(1-1/4 \cdot z^{-1})}{(1-1/2 \cdot z^{-1})(1-1/4 \cdot z^{-1})} \tag{7}$$

which can be rewritten as:

$$\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) Y(z) = \left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) X(z) \tag{8}$$

Transforming back into the time domain yields:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - \frac{7}{12}x(n-1) + \frac{1}{12}x(n-2) \tag{9}$$

Since the system is described by a CCDE and we know that it is linear, its initial conditions must be:

$$y(-2) = y(-1) = 0 \tag{10}$$

(c) If $y(n)$ is applied to the input of the system, what would its response be?
Solution. If the new input to the system is $x'(n) \triangleq y(n) = (1/2)^n u(n)$, we have for the z-transform of the new output $y'(n)$:

$$\begin{aligned} Y'(z) &= H(z)X'(z) \\ &= \frac{(z-1/3)(z-1/4)}{(z-1/2)(z-1/4)} \cdot \frac{z}{z-1/2} \\ &= \frac{z^2}{(z-1/2)^2} - \frac{1}{3}\frac{z}{(z-1/2)^2} \\ &= 2z\frac{1/2 \cdot z}{(z-1/2)^2} - \frac{2}{3}\frac{1/2 \cdot z}{(z-1/2)^2} \end{aligned} \tag{11}$$

Using linearity of the z-transform and Table 9.4 in the course reader, we find the inverse transform to be:

$$
\begin{aligned}
y(n) &= 2(n+1)\left(\frac{1}{2}\right)^{n+1} u(n+1) - \frac{2}{3}n\left(\frac{1}{2}\right)^{n} u(n) \\
&= (n+1)\left(\frac{1}{2}\right)^{n} u(n) - \frac{2}{3}n\left(\frac{1}{2}\right)^{n} u(n) = (\frac{1}{3}n+1)\left(\frac{1}{2}\right)^{n} u(n) \qquad (12)
\end{aligned}
$$

3. (20 PTS) Consider a causal system described by the difference equation

$$y(n) = y(n-1) - \frac{1}{4}y(n-2) + \left(\frac{1}{4}\right)^n u(2n-1), \quad y(0) = a, \quad y(1) = b$$

(a) Find a particular solution to the given system.
Solution. Letting $x(n) \triangleq \left(\frac{1}{4}\right)^n u(2n-1) = \left(\frac{1}{4}\right)^n u(n-1)$, we have

$$
\begin{aligned}
y(n) - y(n-1) + \frac{1}{4}y(n-2) &= x(n) \\
y(0) = a,\ y(1) &= b \qquad (13)
\end{aligned}
$$

Given $x(n) = u(n-1)$, we set $y_p(n) = K\left(\frac{1}{4}\right)^n u(n)$. Then

$$K\left(\frac{1}{4}\right)^n u(n) - K\left(\frac{1}{4}\right)^{n-1} u(n-1) + \frac{1}{4}K\left(\frac{1}{4}\right)^{n-2} u(n-2) = \left(\frac{1}{4}\right)^n u(n-1) \qquad (14)$$

and for $n \geq 2$

$$K - 4K + 4K = 1 \implies K = 1$$

Hence

$$y_p(n) = \left(\frac{1}{4}\right)^n u(n) \qquad (15)$$

(b) Find the complete solution in terms of a and b.
Solution. The homogeneous solution satisfies

$$y_h(n) - y_h(n-1) + \frac{1}{4}y_h(n-2) = 0 \qquad (16)$$

The characteristic polynomial is

$$\lambda^2 - \lambda + \frac{1}{4} = 0 \implies \lambda_{1,2} = \frac{1}{2} \qquad (17)$$

The full solution hence has the form

$$y(n) = y_h(n) + y_p(n) = C_1\left(\frac{1}{2}\right)^n + C_2 \cdot n\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n, \quad n \geq 2 \qquad (18)$$

We determine C_1 and C_2 from the initial conditions:

$$y(0) = a = C_1 + 0 + 1 \implies C_1 = a - 1 \qquad (19)$$

$$y(1) = b = (a-1)\cdot\frac{1}{2} + C_2 \cdot \frac{1}{2} + \frac{1}{4} \implies C_2 = 2b - a + \frac{1}{2} \qquad (20)$$

The full solution is then

$$y(n) = \left[(a-1)\left(\frac{1}{2}\right)^n + \left(2b - a + \frac{1}{2}\right)n\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n) \qquad (21)$$

(c) Given that $2b - a = -\frac{1}{2}$, compute the energy of $y(n)$. For what values of a and b, $y(n)$ has the smallest energy possible?
Solution. Given that $2b - a = -\frac{1}{2}$, the output reduces to the following form:

$$y(n) = \left[(a-1)\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n) \tag{22}$$

The energy of $y(n)$ is thus:

$$\begin{aligned}\mathcal{E}_y &= \sum_n |y(n)|^2 = \sum_n |y(n)^2| \\ &= \sum_{n=0}^{\infty} (a-1)^2 \left(\frac{1}{4}\right)^n + 2(a-1)\left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n \\ &= (a-1)^2 \frac{1}{1-\frac{1}{4}} + 2(a-1)\frac{1}{1-\frac{1}{8}} + \frac{1}{1-\frac{1}{16}} \\ &= (a-1)^2 \frac{4}{3} + (a-1)\frac{16}{7} + \frac{16}{15} \end{aligned} \tag{23}$$

$\mathcal{E}_y$ is a quadratic positive function, to find the a that minimizes it, we compute first its derivative with respect to a and then we set this derivative to zero:

$$\frac{8}{3}(a-1) + \frac{16}{7} = 0$$

Thus $a = 1 - \frac{6}{7} = \frac{1}{7}$ and $b = \frac{a-\frac{1}{2}}{2} = -\frac{5}{28}$.

4. (20 PTS) Consider the following system

$$\begin{aligned} y(n) &= \sum_{k=1}^{n} \lambda^{n-k} x(k)x(k+1), \quad n \geq 1 \\ y(0) &= 0 \end{aligned}$$

where $\lambda \in (0, 1)$.

(a) Find an expression relating $y(n)$ to $y(n-1)$.
Solution. We can write for $n \geq 1$:

$$\begin{aligned} y(n) &= \sum_{k=1}^{n} \lambda^{n-k} x(k)x(k+1) \\ &= \lambda^{n-n} x(n)x(n+1) + \lambda \sum_{k=1}^{n-1} \lambda^{n-1-k} x(k)x(k+1) \\ &= x(n)x(n+1) + \lambda y(n-1) \end{aligned} \tag{24}$$

(b) Is the system causal? linear? stable?
Solution. The system is *not causal*, since the $y(n)$ depends on $x(n+1)$. It is *not linear*. A suitable counterexample are the two-sequences:

$$x_1(n) = \begin{cases} u(n) \text{ for odd } n, \\ 0 \text{ for even } n. \end{cases} \tag{25}$$

$$x_2(n) = \begin{cases} 0 \text{ for odd } n, \\ u(n) \text{ for even } n. \end{cases} \tag{26}$$

Since $x_1(k)x_1(k+1) = x_2(k)x_2(k+1) = 0 \;\forall\; k$, we have $y_1(n) = y_2(n) = 0 \;\forall\; n$, whereas the response $y(n)$ to $x_1(n) + x_2(n) = u(n) \;\forall\; n$ can be verified to satisfy:

$$y(n) = \sum_{k=1}^{n} \lambda^{n-k} \neq 0, \quad n \geq 1 \tag{27}$$

The system is *stable*. Assume $|x(n)| \leq B_x$, then

$$\begin{aligned}
|y(n)| &= \left| \sum_{k=1}^{n} \lambda^{n-k} x(k)x(k+1) \right| \\
&\leq \sum_{k=1}^{n} \left| \lambda^{n-k} x(k)x(k+1) \right| \\
&\leq \sum_{k=1}^{n} \lambda^{n-k} \left| x(k) \right| \left| x(k+1) \right| \\
&\leq \sum_{k=1}^{n} \lambda^{n-k} B_x^2 \\
&\leq B_x^2 \lambda^{n-1} \cdot \sum_{k=0}^{n-1} \left(\lambda^{-1} \right)^k \\
&\leq B_x^2 \lambda^{n-1} \cdot \frac{1 - \lambda^{-n}}{1 - \lambda^{-1}} \\
&= B_x^2 \cdot \frac{\lambda^n - 1}{\lambda - 1} \\
&\leq \infty
\end{aligned} \tag{28}$$

for all n as long as $|\lambda| < 1$, which is the case.

(c) Is the system relaxed? time-invariant?
Solution. The system is *relaxed*. Assume $x(n) = 0$ for $n \leq n_x$. Then $x(k)x(k+1) = 0$ for $k \leq n_x$, which implies $y(n) = 0$ for $n \leq n_x$. It is also *time-invariant*. To see this, note that for $n \geq 1$,

$$\begin{aligned}
y_K(n) &= \sum_{k=1}^{n} \lambda^{n-k} x(k-K)x(k-K+1) \\
&= \sum_{k=K+1}^{n} \lambda^{n-k} x(k-K)x(k-K+1) \\
&= \sum_{k'=1}^{n-K} \lambda^{n-k'-K} x(k')x(k'+1) \\
&= y(n-K)
\end{aligned} \tag{29}$$

where we used the fact that $x(n) = 0$ for $n \leq 0$ since the system runs for $n \geq 1$.