EC ENGR113-1 (2018 Winter) Mid-Term Exam - Solutions

February 12, 2018 (duration: 110 minutes)

Instructions: Attempt all questions but you can drop one of the 10-point problems (i.e. 5 out of the 6 ten-point problems will count towards your final score). Show all your work! Put your name on each answer sheet and number all pages. Turn in your answer sheets stapled with this question paper on top.

1. (5 points) Determine whether or not the following signals are periodic. If periodic, determine the fundamental period.
   a) $x[n] = \cos\left(\frac{3\pi n}{11} + 1\right)$
   b) $x[n] = e^{\frac{j\pi n}{4}}$
   c) $x[n] = \sin\left(\frac{3n}{4}\right)$

a) Discrete sequence $x[n]$ is periodic if and only if there exists a positive integer N such that $x[n+N] = x[n]$, i.e. the sequence repeats itself after N samples.

$$\Rightarrow \cos\left(\frac{3\pi(n+N)}{11} + 1\right) = \cos\left(\frac{3\pi n}{11} + 1\right)$$

$$\Rightarrow \frac{3\pi(n+N)}{11} + 1 = \left(\frac{3\pi n}{11} + 1\right) + 2\pi k \quad \text{(for some integer } k\text{)}$$

$$\Rightarrow 3\pi N = 22\pi k$$

$$\Rightarrow N = 22\frac{k}{3}$$

So if we choose the smallest value for the arbitrary integer k to be $k = 3$, we can find the smallest value of the integer period, N, to be $N = 22$ for which the definition of periodicity for $x[n]$ is true. That is, the signal is periodic with a fundamental period of 22.

b) Same definition and method as used for part (a) above. We need to determine smallest integer N, if it exists such that:

$$e^{\frac{j\pi(n+N)}{4}} = e^{\frac{j\pi n}{4}}$$

$$\Rightarrow \frac{\pi(n+N)}{4} = \frac{\pi n}{4} + 2\pi k$$

$$\Rightarrow N = 8k$$

So if we choose $k = 1$, we get the smallest integer value of $N = 8$. So the sequence is indeed periodic with fundamental period $N = 8$

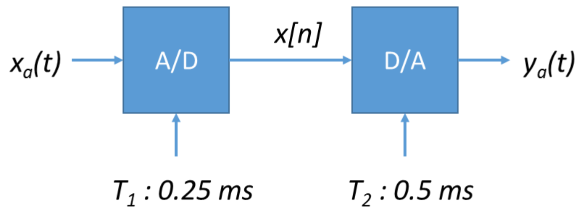
c) Again as for parts (a) and (b), we need to determine smallest integer N, if it exists, such that:

$$\sin\left[\frac{3(n+N)}{4}\right] = \sin\left[\frac{3n}{4}\right]$$

$$\Rightarrow \frac{3(n+N)}{4} = \frac{3n}{4} + 2\pi k$$

$$\Rightarrow N = \frac{8}{3}\pi k$$

But in this case, note that no choice for the integer k can give an integer value for N. So in this case, the sequence cannot be periodic, and there is no fundamental period



2. (10 pts) Given $x_a(t) = 2\sin(7200\pi t)$ in the system shown on the right. Find $y_a(t)$.

$x[n] = x_a(t)|_{t=nT_1} = 2\ \sin(7200\pi n \cdot 0.25 \cdot 10^{-3})$
$x[n] = 2\sin[1.8\pi n]$

So the output of the A/D is the discrete sinusoid $x[n]$ which has frequency $\omega = 1.8\pi$. But note that these are aliased frequencies ($\omega > \pi$), so that the resulting discrete sequence is indistinguishable from a "folded" sinusoid at lower frequency. This lower frequency (in the usual range $-\pi \leq \ \omega \leq \pi$) can be obtained by subtracting as many multiples of π as necessary to bring it in the usual range. So:

$x[n] = 2\sin[1.8\pi n] = 2\sin[1.8\pi n - 2\pi n]$
$x[n] = 2\ \sin[-0.2\pi n]$
$x[n] = -2\ \sin[0.2\pi n]$

This sequence is then reconstructed as an ideal sinusoidal analog signal with D/A clock outputting and a value $x[n]$ for every $n = \frac{t}{T_2}$ instants and shaping the resulting analog waveform. So:

$y_a(t) = x[n]|_{n=\frac{t}{T_2}} = -2\sin(0.2\pi t \cdot 2000)$
$y_a(t) = -2\sin(400\pi t)$

3. (10 pts) Consider the discrete-time system described by the following input-output relationship:

$$y[n] = \frac{x[n] + x[-n]}{2}$$

Analyze and determine the system for each of the following properties: ***a)*** static or dynamic, ***b)*** linear or non-linear, ***c)*** time invariant or time varying, ***d)*** causal or non-causal, ***e)*** stable or unstable. Show all your work!

a) To be static, $y[n]$ can depend only on the present input. However in this case, $y[n]$ also needs input values at minus n. So the system cannot be static, therefore it is dynamic

b) Applying definition of linearity; let $y_1[n] = \frac{x_1[n]+x_1[-n]}{2}$, $y_2[n] = \frac{x_2[n]+x_2[-n]}{2}$
Then we need to show that for combined input $\alpha x_1[n] + \beta x_2[n]$, the output must be $y[n] = \alpha y_1[n] + \beta y_2[n]$.

$$y[n] = \frac{\alpha x_1[n] + \beta x_2[n]}{2} + \frac{\alpha x_1[-n] + \beta x_2[-n]}{2}$$
$$y[n] = \alpha \frac{x_1[n] + x_1[-n]}{2} + \beta \frac{x_2[n] + x_2[-n]}{2}$$
$$\Rightarrow y[n] = \alpha y_1[n] + \beta y_2[n]$$

Therefore the system is indeed linear

c) For time invariance, we need to check outputs for input $x[n]$ and time shifted input $x[n-a]$ to check if the output is similarly shifted. So, for $x_1[n]$ input, if output is $y_1[n]$, then for $x_1[n-a]$ input we need to show that the output is:

$$y_1[n-a] = \frac{\mathrm{x}[n-a] + \mathrm{x}[-(n-a)]}{2} = \frac{\mathrm{x}[n-a] + \mathrm{x}[-n+a]}{2}$$

But we see that if $x[n] = x_1[n-a]$ is a input into the system, what we actually get is:

$$y[n] = \frac{\mathrm{x}[n] + \mathrm{x}[-n]}{2} = \frac{\mathrm{x}_1[n-a] + \mathrm{x}_1[(-n)-a]}{2} = \frac{\mathrm{x}_1[n-a] + \mathrm{x}_1[-n-a]}{2}$$

Since $x[m] = x_1[m-a]$ and we substitute $m = n$ for the first term above to get $x[n] = x_1[n-a]$, we substitute $m = -n$ for the second term to get $x[-n] = x_1[(-n)-a]$. Therefore $x_1[n-1] \nrightarrow y_1[n-a]$ so the system is time varying.

d) For causality, the output at any instant must only depend on the present and past time values for all integer n. But we see that, e.g. when $n = -3$, $y[-3]$ depends on $x[-(-3)] = x[3]$, that is output at a particular instant $(n = -3)$ depends on a future value of the input. Hence the system is non-causal.

3) For stability, we need to check if for all bounded inputs whether the output is bounded or not. If the input is bounded, it means $|x[n]| \le M < \infty$ for all n. Therefore:

$$|y[n]| = \left|\frac{x[n] + x[-n]}{2}\right|$$
$$|y[n]| \le \left|\frac{x[n]}{2}\right| + \left|\frac{x[-n]}{2}\right| = \frac{|x[n]|}{2} + \frac{|x[-n]|}{2}$$
$$|y[n]| \le \frac{M}{2} + \frac{M}{2} = M < \infty$$

Therefore the system is stable

4. a) (3 pts) If $h[n]$ is the impulse response of a linear time invariant system, and $s[n] = h[n] * u[n]$ is the step response the system ($u[n]$ is the step input), show that $h[n] = s[n] - s[n-1]$.
b) (7 pts) Given $h[n] = a^n u[n]$, where $0 < a < 1$, find the closed form expression (without using z-transforms) for the step response $s[n]$ of the system.

a) Note that for an LTI system $s[n-1] = h[n] \star u[n-1]$. So:

$\delta[n] = u[n] - u[n-1]$
$\delta[n] \star h[n] = (u[n] - u[n-1]) \star h[n]$
$h[n] = u[n] \star h[n] - u[n-1] \star h[n]$
$h[n] = s[n] - s[n-1]$

b) Note that $h[n] = \{1, a, a^2, a^3, \cdots\}$. If the input to this system the step function $x[n] = u[n]$, then by straightforward convolution operation (flip and slide the $x[n]$ step input from the left), we see that $y[n] = 0$ for $n < 0$ and for $n > 0$:

$y[0 = 1$
$y[1] = 1 + a$
$y[2] = 1 + a + a^2$
$\vdots$
$y[n] = 1 + a + a^2 + a^3 + \cdots + a^n$

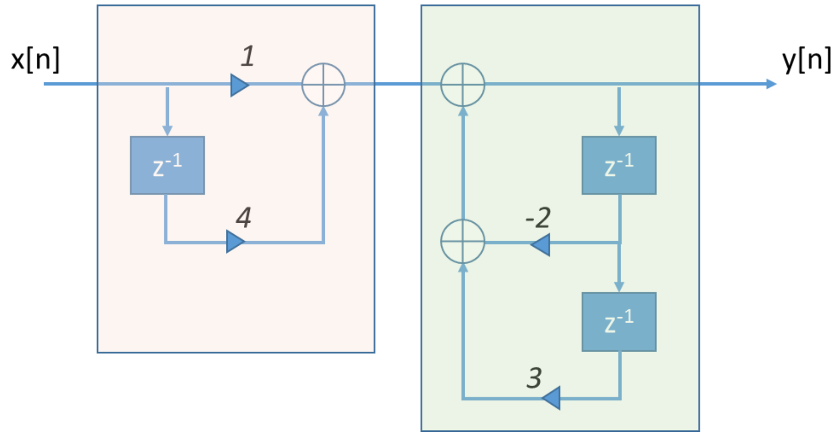
This is simply a geometric series (which also converges since it is given that $0 < a < 1$), and thus:

$$y[n] = \frac{1 - a^{n+1}}{1 - a}$$

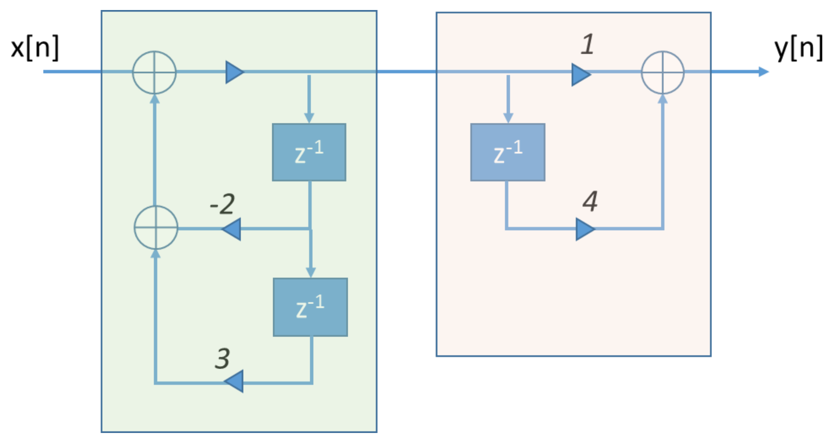
5. (5 pts) Determine and draw the Direct Form II realization of the following LTI system:

$$y[n] + 2y[n-1] - 3y[n-2] = x[n] + 4x[n-1]$$

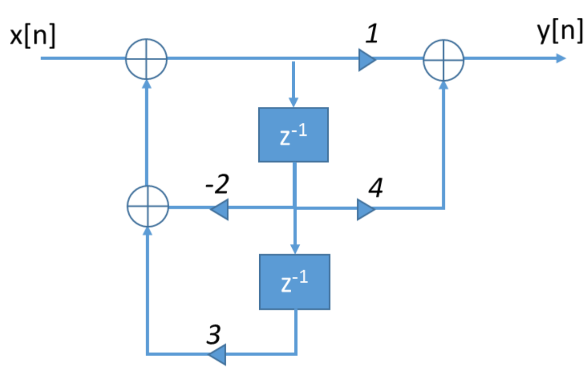
The left hand and right hand sides of the equation are converted into the shaded structures shown below.



However since the system is an LTI structure, the shaded blocks can be interchanged as follows:



The delay elements can now be combined to produce the fewest memory element Direct Form II structure shown below:



6. (10 pts) Find the z-transforms for the following (including region of convergence):

a) $x[n] = \left(\frac{1}{3}\right)^n u[n-3]$

b) $x[n] = \left(\frac{1}{2}\right)^{|n|}$

c) $x[n] = \sin[\omega n]\, u[n]$

a) $x[n] = \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^{n-3} u[n-3]$ which is of the form $x[n] = \left(\frac{1}{3}\right)^3 p[n-3]$ where $p[n-3]$ is a time shifted version of $p[n] = \left(\frac{1}{3}\right)^n u[n]$. Note that the z-transform of $p[n]$ is simply $\frac{1}{1-\frac{1}{3}z^{-1}}$ with region of convergence $|z| > \frac{1}{3}$. So, by the time shifting property:

$$X(z) = \left(\frac{1}{3}\right)^3 z^{-3} P[z]$$

$$X(z) = \left(\frac{1}{3}\right)^3 z^{-3} \frac{1}{1-\frac{1}{3}z^{-1}} \quad \text{with ROC: } |z| > \frac{1}{3}$$

b) Use z-transform definition and split the summation into two halves:

$$X(z) = X_1(z) + X_2(z) = \sum_{n=-\infty}^{-1} \frac{1}{2}^{-n} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{2}^{n} z^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{2}^{n} z^{n} + \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{with } |z| > \frac{1}{2}$$

$$X(z) = \frac{\frac{1}{2}z}{\left(1-\frac{1}{2}z\right)} + \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} \quad \text{with } \frac{1}{2} < |z| < 2$$

$$X(z) = -\frac{3z}{(2z-1)(z-2)} \quad \text{ROC: } \frac{1}{2} < |z| < 2$$

c) Use Euler's formula:

$$x[n] = \frac{1}{2j}\left(e^{j\omega n} - e^{-j\omega n}\right)u[n]$$

$$X(z) = \frac{1}{2j}\left(\frac{1}{1-e^{j\omega}z^{-1}} - \frac{1}{(1-e^{-j\omega}z^{-1}}\right)$$

$$X(z) = \frac{\sin(\omega)\, z^{-1}}{1 - 2\cos(\omega) z^{-1} + z^{-2}} \quad \text{with ROC: } |z| > 1$$

7. (10 pts) The input sequence $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$ is passed through the digital filter described by the difference equation: $y[n] = x[n] + \frac{1}{4}y[n-2]$. Find the output of the filer $y[n]$ in the usual closed form (use z-transforms).

The approach is to take z-transforms of input and transfer function, multiply in the z-domain to get the output ($Y(z) = H(z) \cdot X(z)$) and then invert to get the output sequence equation.

$$x[n]x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \ \Rightarrow \ X(z) = 1 + 2z^{-1} + z^{-2}$$

$$y[n] = x[n] + \frac{1}{4}y[n-2] \Rightarrow Y(z) = X(z) + \frac{1}{4}z^{-2}Y(z)$$

$$\Rightarrow \left(1 - \frac{1}{4}z^{-2}\right)Y(z) = X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{4}z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

$$\therefore Y[z] = H(z) \cdot X(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

Use partial fraction expansion to get:

$$Y[z] = -4 + \frac{5 + 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

$$= -4 + \frac{9}{2\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{1}{2\left(1 + \frac{1}{2}z^{-1}\right)}$$

Using z-transform inversion table:

$$y[n] = -4\delta[n] + \frac{9}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n]$$

8. (10 pts) Determine the Fourier transform of the discrete sequence $x[n] = a^{|n|}$, where $|a| < 1$

Using the definition of the Fourier transform: $x(\omega) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega k}$ and splitting $x[n]$ into the $n < 0$ and $n \geq 0$ domains, we get

$$X(\omega) = X_1(\omega) + X_2(\omega) = \sum_{k=-\infty}^{-1} a^{-k}e^{-j\omega k} + \sum_{k=0}^{\infty} a^{k}e^{-j\omega k}$$

$X_2(\omega)$ is a straightforward infinite geometric sum which, for $|a| < 1$, converges to $X_2(\omega) = \frac{1}{1-ae^{-j\omega}}$
$X_1(\omega)$ can be simplified to form another geometric sum by changing the index of summation and sign of the powers of the terms:

$$X_1(\omega) = \sum_{k=-\infty}^{-1} a^{-k}e^{-j\omega k} = \sum_{k=1}^{\infty} a^{k}e^{j\omega k} = ae^{j\omega k} + a^2e^{j\omega 2k} + \cdots$$

This is an infinite geometric sum with first term $ae^{j\omega k}$ which also converges for $|a| < 1$ to:

$$X_1(\omega) = \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

Combining $X_1(\omega) + X_2(\omega)$:

$$X(\omega) = \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}}$$

$$= \frac{ae^{j\omega}\left(1 - ae^{-j\omega}\right) + 1 - ae^{j\omega}}{(1 - ae^{j\omega})(1 - ae^{-j\omega})}$$

$$= \frac{-a^2e^{j\omega - j\omega} + ae^{j\omega} - ae^{j\omega} + 1}{1 + -a^2e^{j\omega - j\omega} - a(e^{j\omega} + e^{-j\omega})}$$

$$\therefore X(\omega) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$