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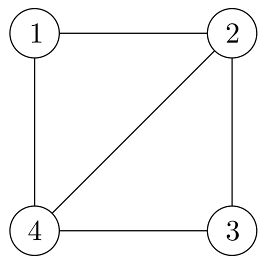
EE103

Midterm solutions

Problem 1 (20 points). We consider an undirected graph with n vertices (nodes), numbered 1 to n. For every pair of vertices i, j with $i \neq j$, there is at most one edge (link) connecting the two vertices. The figure shows a simple example with $n = 4$.



The *Laplacian matrix* associated with the graph is a symmetric $n \times n$-matrix A defined as follows. The ith diagonal entry A_{ii} is equal to the number of edges connected to vertex i (this number is known as the *degree* of vertex i). The off-diagonal entries A_{ij}, $i \neq j$, are given by

$$A_{ij} = \begin{cases} -1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{if vertices } i \text{ and } j \text{ are not connected by an edge.} \end{cases}$$

The Laplacian matrix for the graph in the example is

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

1. (10 points) Show that the Laplacian matrix A in the example is positive semidefinite.
2. (10 points) Show that this property holds in general: the Laplacian matrix of any undirected graph is positive semidefinite. Can a Laplacian matrix be positive definite?

Solution.

1. We show that $x^T Ax \geq 0$ for all x:

$$\begin{aligned} x^T Ax &= 2x_1^2 - 2x_1x_2 - 2x_1x_4 + 3x_2^2 - 2x_2x_3 - 2x_2x_4 + 2x_3^2 - 2x_3x_4 + 3x_4^2 \\ &= (x_1 - x_2)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_4)^2 \\ &\geq 0. \end{aligned}$$

2. In general, if we denote by d_i the degree of vertex i,

$$\begin{aligned} x^T Ax &= \sum_{i=1}^{n} A_{ii}x_i^2 + 2\sum_{i<j} A_{ij}x_ix_j \\ &= \sum_{i=1}^{n} d_i x_i^2 - \sum_{\substack{\text{edges } (i,j) \\ i<j}} 2x_ix_j \\ &= \sum_{\substack{\text{edges } (i,j) \\ i<j}} (x_i - x_j)^2 \\ &\geq 0. \end{aligned}$$

A is not positive definite because $x^T Ax = 0$ for $x = (1, 1, \ldots, 1)$. One can also observe directly that $Ax = 0$ for $x = (1, 1, \ldots, 1)$ because in each row of A the diagonal entry (the degree of the vertex) is equal to the number of entries equal to -1.

Problem 2 (20 points). Let U and V be orthogonal matrices ($U^TU = I$ and $V^TV = I$) of size $m \times n$ with $m > n$, and define $A = UV^T$. For each of the following three statements, either show that it is true, or give a small example (i.e., a specific U, V) for which it is false.

1. (6 points) A is nonsingular.
2. (7 points) A is orthogonal.
3. (7 points) $\|A\| = 1$.

Solution.

1. False. Take for example $m = 2$, $n = 1$,

$$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. False. Same example.
3. True. We have

$$\begin{aligned} \|A\|^2 = \max_{x\neq 0} \frac{\|Ax\|^2}{\|x\|^2} &= \max_{x\neq 0} \frac{x^TVU^TUV^Tx}{\|x\|^2} \\ &= \max_{x\neq 0} \frac{x^TVV^Tx}{\|x\|^2} \\ &= \max_{x\neq 0} \frac{\|V^Tx\|^2}{\|x\|^2} \\ &= \|V^T\|^2 \\ &= 1. \end{aligned}$$

The last line follows because V is orthogonal, so $\|V^T\| = \|V\| = 1$.

Problem 3 (20 points). Let A be a positive definite $n \times n$ matrix. Explain how you can solve each subproblem using a Cholesky factorization of A. Carefully explain the different steps in your algorithm and the cost of each step (number of flops for large n). If you know several methods, give the most efficient one (least number of flops for large n).

1. (6 points) Compute

$$c^T A^{-1} B B^T A^{-1} c$$

where B is an $n \times n$-matrix and c is an n-vector.

2. (7 points) Solve the equation

$$\begin{bmatrix} A & -A \\ -A & 2A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}.$$

The n-vectors b and c are given. The variables are the two n-vectors x, y.

3. (7 points) Solve the equation

$$\begin{bmatrix} A & 0 & u \\ 0 & A & v \\ u^T & v^T & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}.$$

The variables are the n-vectors x, y, and the scalar z. The n-vectors b, c, u, v, and the scalar d are given.

Solution. Let $A = LL^T$ be the Cholesky factorization of A (computed at a cost of $(1/3)n^3$ flops).

1. (a) Solve $LL^T y = c$ by forward and backward substitution ($2n^2$ flops).
   (b) Make the product $z = B^T y$ ($2n^2$ flops).
   (c) Make the product $z^T z$ ($2n$ flops).

   The total is $(1/3)n^3 + 6n^2$ plus lower order terms.

2. We can write the equations as

$$A(x - y) = b, \qquad A(-x + 2y) = c.$$

   Adding the two equations gives $Ay = b + c$. Multiplying the first equation with two and adding it to the second gives $Ax = 2b + c$.

   (a) Solve $LL^T u = b$ ($2n^2$ flops).
   (b) Solve $LL^T v = c$ ($2n^2$ flops).
   (c) Compute $x = 2u + v$ and $y = u + v$ ($3n$ flops).

   The total is $(1/3)n^3 + 4n^2$ plus lower order terms.

3. Use the first two equations to express x and y in terms of z:

$$x = A^{-1}b - zA^{-1}u, \qquad y = A^{-1}c - zA^{-1}v.$$

Substituting in the third equation gives an equation in z:

$$(u^T A^{-1}u + v^T A^{-1}v + 1)z = -d + u^T A^{-1}b + v^T A^{-1}c.$$

(a) Use forward substitution to compute

$$\tilde{b} = L^{-1}b, \qquad \tilde{u} = L^{-1}u, \qquad \tilde{c} = L^{-1}c, \qquad \tilde{v} = L^{-1}v$$

($4n^2$ flops).

(b) Compute

$$z = \frac{-d + \tilde{u}^T\tilde{b} + \tilde{v}^T\tilde{c}}{1 + \tilde{u}^T\tilde{u} + \tilde{v}^T\tilde{v}}$$

($8n$ flops).

(c) Use back substitution to compute

$$x = L^{-T}(\tilde{b} - z\tilde{u}), \qquad y = L^{-T}(\tilde{c} - z\tilde{v})$$

($2n^2$ flops).

The total is $(1/3)n^3 + 6n^2$ plus lower order terms.

Problem 4 (20 points). Suppose A and B are nonsingular $n \times n$-matrices, and that $A + B$ is also nonsingular.

1. (10 points) Show that $A^{-1} + B^{-1}$ is nonsingular with inverse

$$(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B.$$

2. (10 points) Use the expression in part 1 to derive an algorithm for solving the equation

$$(A^{-1} + B^{-1})x = b,$$

using a single LU factorization of size $n \times n$. The matrices A, B and the n-vector b are given. Carefully explain the different steps in your algorithm and the complexity of each step (number of flops for large n). If you know several methods, give the most efficient one.

Solution.

1. We verify that $(A^{-1}+B^{-1})A(A+B)^{-1}B = I$.

$$\begin{aligned}
(A^{-1}+B^{-1})(A(A+B)^{-1}B) &= (I+B^{-1}A)(A+B)^{-1}B \\
&= B^{-1}(B+A)(A+B)^{-1}B \\
&= B^{-1}B \\
&= I.
\end{aligned}$$

   Another proof:

$$(A^{-1}+B^{-1})^{-1} = (B^{-1}(B+A)A^{-1})^{-1} = A(B+A)^{-1}B.$$

2. Using the expression for the inverse the solution x can be written as

$$x = (A^{-1}+B^{-1})^{-1}b = A(A+B)^{-1}Bx.$$

   This can be computed using an LU factorization of $A+B$.

   (a) Compute $y = Bx$ ($2n^2$ flops).
   (b) Compute $C = A+B$ (n^2 flops) and solve $Cz = y$ ($(2/3)n^3 + 2n^2$).
   (c) Compute $x = Cz$ ($2n^2$ flops).

   The total is $(2/3)n^3$ plus lower order terms.