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Midterm Solutions

Problem 1 (10 points). The *cross-product* $a \times x$ of two 3-vectors $a = (a_1, a_2, a_3)$ and $x = (x_1, x_2, x_3)$ is defined as the vector

$$a \times x = \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix}.$$

1. Assume a is fixed and nonzero. Show that the function $f(x) = a \times x$ is a linear function of x, by giving a matrix A that satisfies $f(x) = Ax$ for all x.
2. Is the matrix A you found in part 1 singular or nonsingular?
3. Verify that $A^TA = (a^Ta)I - aa^T$.
4. Use the observations in parts 1 and 3 to show that for nonzero x,

$$\|a \times x\| = \|a\|\,\|x\|\,|\sin\theta|$$

   where θ is the angle between a and x.

Solution.

1. $a \times x = Ax$ with
$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$
2. Singular, because $Aa = 0$ so A has a nonzero vector in its nullspace.
3. Working out the matrix product gives

$$\begin{aligned} A^TA &= \begin{bmatrix} a_2^2 + a_3^2 & -a_1a_2 & -a_1a_3 \\ -a_1a_2 & a_1^2 + a_3^2 & -a_2a_3 \\ -a_1a_3 & -a_2a_3 & a_1^2 + a_3^2 \end{bmatrix} \\ &= \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & 0 & 0 \\ 0 & a_1^2 + a_2^2 + a_3^2 & 0 \\ 0 & 0 & a_1^2 + a_2^2 + a_3^2 \end{bmatrix} - \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} \\ &= (a^Ta)I - aa^T. \end{aligned}$$

4. From the expression in part 3,

$$
\begin{aligned}
\|a \times x\|^2 &= x^T A^T A x \\
&= x^T((a^T a)I - aa^T)x \\
&= (a^T a)(x^T x) - (x^T a)^2 \\
&= \|a\|^2 \|x\|^2 - (\|a\| \|x\| \cos\theta)^2 \\
&= (\|a\| \|x\| \sin\theta)^2 .
\end{aligned}
$$

Problem 2 (15 points). Let A be defined as

$$A = I + BB^T$$

where B is a given orthogonal $n \times m$ matrix (not necessarily square).

1. Show that A is positive definite.
2. What is the cost (number of flops for large m and n) of solving $Ax = b$ by first computing $A = I + BB^T$ and then applying the standard method for linear equations $Ax = b$ with positive definite A?
3. Show that $A^{-1} = I - (1/2)BB^T$.
4. Use the expression in part 3 to derive an efficient method for solving $Ax = b$ (*i.e.*, a method that is much more efficient than the method in part 2.) Give the cost of your method (number of flops for large m and n).

Solution.

1. We verify the definition of positive definite matrix:

   $$x^T Ax = x^T x + x^T BB^T x = \|x\|^2 + \|B^T x\|^2.$$

   This is clearly positive for all nonzero x.

2. Computing A takes $mn^2 + n$ flops (mn^2 for the symmetric matrix product $B^T B$ and n for adding the identity matrix). The Cholesky factorization costs $(1/3)n^3$. Forward and backward substitution cost $2n^2$. The total is $(1/3)n^3 + mn^2$.

3. We show that $AA^{-1} = I$:

   $$(I + BB^T)(I - \frac{1}{2}BB^T) = I - \frac{1}{2}BB^T + BB^T - \frac{1}{2}BB^T BB^T = I.$$

   In the last step we use the definition of orthogonal matrix ($B^T B = I$).

4. We can solve $Ax = b$ by evaluating $x = A^{-1}b$ using the formula in part 3:

   $$x = (I - \frac{1}{2}BB^T)b = b - \frac{1}{2}B(B^T b).$$

   This costs $4mn$ for the matrix vector products $y = B^T x$ and By, and $2n$ for the scalar multiplication and addition. The total is $6mn$.

Problem 3 (10 points). Let A be a positive definite matrix of size $n \times n$. For what values of the scalar a are the following matrices positive definite?

$$\text{(a)} \quad \left[\begin{array}{cc} A & ae_1 \\ ae_1^T & 1 \end{array}\right] \qquad \text{(b)} \quad \left[\begin{array}{cc} A & e_1 \\ e_1^T & a \end{array}\right] \qquad \text{(c)} \quad \left[\begin{array}{cc} A & ae_1 \\ ae_1^T & a \end{array}\right].$$

($e_1 = (1, 0, \ldots, 0)$ denotes the first unit vector of length n.) Give your answer for each of the three problems in the form of upper and/or lower bounds ($a_{\min} < a < a_{\max}$, $a > a_{\min}$, or $a < a_{\max}$). Explain how you can compute the limits $a_{\min}$ and $a_{\max}$ using the Cholesky factorization $A = LL^T$. (You don't need to discuss the complexity of the computation.)

Solution. The Cholesky factorizations of the three matrices (if they exist) are

$$\left[\begin{array}{cc} A & ae_1 \\ ae_1^T & 1 \end{array}\right] = \left[\begin{array}{cc} L & 0 \\ ae_1^T L^{-T} & \sqrt{1 - a^2 e_1^T L^{-T} L^{-1} e_1} \end{array}\right] \left[\begin{array}{cc} L^T & aL^{-1}e_1 \\ 0 & \sqrt{1 - a^2 e_1^T L^{-T} L^{-1} e_1} \end{array}\right]$$

$$\left[\begin{array}{cc} A & e_1 \\ e_1^T & a \end{array}\right] = \left[\begin{array}{cc} L & 0 \\ e_1^T L^{-T} & \sqrt{a - e_1^T L^{-T} L^{-1} e_1} \end{array}\right] \left[\begin{array}{cc} L^T & L^{-1}e_1 \\ 0 & \sqrt{a - e_1^T L^{-T} L^{-1} e_1} \end{array}\right]$$

$$\left[\begin{array}{cc} A & ae_1 \\ ae_1^T & a \end{array}\right] = \left[\begin{array}{cc} L & 0 \\ ae_1^T L^{-T} & \sqrt{a - a^2 e_1^T L^{-T} L^{-1} e_1} \end{array}\right] \left[\begin{array}{cc} L^T & aL^{-1}e_1 \\ 0 & \sqrt{a - a^2 e_1^T L^{-T} L^{-1} e_1} \end{array}\right]$$

This gives the following conditions.

1. Positive definite for $|a| < 1/\sqrt{e_1^T L^{-T} L^{-1} e_1} = 1/\|L^{-1}e_1\|$.
2. Positive definite for $a > e_1^T L^{-T} L^{-1} e_1 = \|L^{-1}e_1\|^2$.
3. Positive definite for $0 < a < 1/(e_1^T L^{-T} L^{-1} e_1) = 1/\|L^{-1}e_1\|^2$.

In each case, we can compute the limits from the norm of the vector $L^{-1}e_1$, which can be computed by solving $Lx = e_1$ via forward substitution.

Problem 4 (15 points). Explain how you can solve the following problems using a single LU factorization and without computing matrix inverses. The matrix A is a given nonsingular $n \times n$ matrix. For each problem, carefully explain the different steps in your algorithm, give the cost of each step, and the total cost (number of flops for large n, excluding the LU factorization itself). If you know several methods, give the most efficient one.

1. Compute $\left(A^{-1} + A^{-T}\right)^2 b$, where b is a given n-vector.
2. Solve the equation $AXA = B$ for the unknown X, where B is a given $n \times n$-matrix.
3. Compute $\sum_{i=1}^{n} \sum_{j=1}^{n} (A^{-1})_{ij}$, the sum of all the entries of A^{-1}.

Solution.

1. Expanding the product gives

$$(A^{-1} + A^{-T})(A^{-1} + A^{-T})b.$$

   This can be computed as follows.

   - Compute $x = A^{-1}b$ by solving $PLUx = b$ ($2n^2$ flops).
     - Solve $Px_1 = b$ (0 flops)
     - Solve $Lx_2 = x_1$ by forward substitution (n^2 flops).
     - Solve $Ux = x_2$ by back substitution (n^2 flops).
   - Compute $y = A^{-T}b$ by solving $U^T L^T P^T y = b$ ($2n^2$ flops).
     - Solve $U^T y_1 = b$ by forward substitution (n^2 flops)
     - Solve $L^T y_2 = y_1$ by back substitution (n^2 flops).
     - Solve $P^T y = y_2$ (0 flops).
   - Compute $z = x + y$ (n flops).
   - Compute $v = A^{-1}z$ by solving $PLUv = z$ as above ($2n^2$ flops).
   - Compute $w = A^{-T}z$ by solving $U^T L^T P^T w = z$ as above ($2n^2$ flops).
   - The final result is $v + w$ (n flops).

   The total is $8n^2$ plus lower order terms.

2. We first solve $AY = B$, column by column. We then solve $A^T Z = Y^T$ column by column. The solution Z is the transpos of X.

   - Solve $PLUY = B$, column by column as explained in part 1 ($2n^3$ flops).
   - Solve $U^T L^T P^T Z = Y^T$, column by column as explained in part 1 ($2n^3$ flops).

The total is $4n^3$ flops.

3. We can write this as $x^T A^{-1} x$ where $x = (1, 1, \ldots, 1)$ is the n-vector with entries equal to one.

   - Compute $y = A^{-1}x$ by solving $PLUy = x$ ($2n^2$ flops).
   - Compute $\sum_i y_i$ (n flops).

   The total is $2n^2$ flops.