

UCLA — Electrical Engineering Dept.  
EE102: Systems and Signals — Midterm Exam  
Wednesday, October 28, 2015

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This exam has 4 questions, for a total of 33 points.

Closed book. One two-sided cheat-sheet allowed.  
Answer the questions in the spaces provided on the question sheets. If you run  
out of room for an answer, continue on the back of the page.  
**Please, write your name and ID on the top of each loose sheet!**

Name and ID: [REDACTED]

Name of person on your left: [REDACTED]

Name of person on your right: [REDACTED]

Question	Points	Score
1	9	8
2	10	10
3	8	7.5
4	6	3.5
Total:	33	29

Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

$$K_1 x_1(t) + K_2 x_2(t) \\ \sin(K_1 x_1(t) + K_2 x_2(t))$$

Non linear

$$z(t) = x(t - t_0)$$

$$T[z(t)] = \sin(z(t+1))$$

$$= \sin(x(t+1 - t_0))$$

What are its properties?

- Linear
- Time-invariant
- Causal

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} e^t e^{-\tau} x(\tau) u(t-\tau) d\tau.$$

$$\delta(t-\sigma)$$

$$[\delta(t-\sigma)]$$

$$= \int_{-\infty}^{\infty} e^t e^{-\tau} \delta(\tau-\sigma)$$

What are its properties?

- Linear
- Time-invariant
- Causal

$$y(t) = \int_{-\infty}^{\infty} e^t e^{-\tau} x(\tau) u(t-\tau) d\tau$$

$$h(t; \tau) = e^t e^{-\tau} u(t-\tau)$$

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

What is the condition on  $a$  for the system to be causal?

- $a < 1$
- $a = 1$
- $a > 1$

$$x(0.5t)$$

$$t = -1$$

$$t = -0.5$$

$$x(-0.5)$$

$$x(-0.25)$$

$$x(t)$$

$$T[\delta(t-\sigma)] = \int_{-\infty}^{\infty} e^t e^{-\tau} \delta(\tau-\sigma) u(t-\tau) d\tau$$

$$z(t) = x(t-t_0) = e^t e^{-\sigma} u(t-\sigma)$$

$$y(t-t_0) = \int_{-\infty}^{\infty} e^{t-t_0} e^{-\tau} x(\tau) u(t-t_0-\tau) d\tau$$

$$T[z(t)] = \int_{-\infty}^{\infty} e^t e^{-\tau} z(\tau) u(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^t e^{-\tau} x(\tau-t_0) u(t-\tau) d\tau$$

The following is a list of the most common
  $\text{Laplace}^{-1}\{F(s)\}$

1.  $\text{Laplace}^{-1}\left\{\frac{1}{s}\right\} = 1$   
 2.  $\text{Laplace}^{-1}\left\{\frac{1}{s^2}\right\} = t$   
 3.  $\text{Laplace}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$   
 4.  $\text{Laplace}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}t^3$   
 5.  $\text{Laplace}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{24}t^4$   
 6.  $\text{Laplace}^{-1}\left\{\frac{1}{s^6}\right\} = \frac{1}{120}t^5$   
 7.  $\text{Laplace}^{-1}\left\{\frac{1}{s^7}\right\} = \frac{1}{5040}t^6$   
 8.  $\text{Laplace}^{-1}\left\{\frac{1}{s^8}\right\} = \frac{1}{40320}t^7$   
 9.  $\text{Laplace}^{-1}\left\{\frac{1}{s^9}\right\} = \frac{1}{362880}t^8$   
 10.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{10}}\right\} = \frac{1}{3628800}t^9$   
 11.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{11}}\right\} = \frac{1}{39916800}t^{10}$   
 12.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{12}}\right\} = \frac{1}{479001600}t^{11}$   
 13.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{13}}\right\} = \frac{1}{635136000}t^{12}$   
 14.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{14}}\right\} = \frac{1}{8871680000}t^{13}$   
 15.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{15}}\right\} = \frac{1}{133075200000}t^{14}$   
 16.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{16}}\right\} = \frac{1}{2016115200000}t^{15}$   
 17.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{17}}\right\} = \frac{1}{30241728000000}t^{16}$   
 18.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{18}}\right\} = \frac{1}{443424384000000}t^{17}$   
 19.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{19}}\right\} = \frac{1}{6651365760000000}t^{18}$   
 20.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{20}}\right\} = \frac{1}{100000000000000000}t^{19}$   
 21.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{21}}\right\} = \frac{1}{1500000000000000000}t^{20}$   
 22.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{22}}\right\} = \frac{1}{22000000000000000000}t^{21}$   
 23.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{23}}\right\} = \frac{1}{330000000000000000000}t^{22}$   
 24.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{24}}\right\} = \frac{1}{4840000000000000000000}t^{23}$   
 25.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{25}}\right\} = \frac{1}{72600000000000000000000}t^{24}$   
 26.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{26}}\right\} = \frac{1}{1089000000000000000000000}t^{25}$   
 27.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{27}}\right\} = \frac{1}{16335000000000000000000000}t^{26}$   
 28.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{28}}\right\} = \frac{1}{245016000000000000000000000}t^{27}$   
 29.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{29}}\right\} = \frac{1}{3675240000000000000000000000}t^{28}$   
 30.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{30}}\right\} = \frac{1}{55128600000000000000000000000}t^{29}$   
 31.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{31}}\right\} = \frac{1}{826929000000000000000000000000}t^{30}$   
 32.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{32}}\right\} = \frac{1}{12403935000000000000000000000000}t^{31}$   
 33.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{33}}\right\} = \frac{1}{186059025000000000000000000000000}t^{32}$   
 34.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{34}}\right\} = \frac{1}{2790871500000000000000000000000000}t^{33}$   
 35.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{35}}\right\} = \frac{1}{41863072500000000000000000000000000}t^{34}$   
 36.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{36}}\right\} = \frac{1}{627946050000000000000000000000000000}t^{35}$   
 37.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{37}}\right\} = \frac{1}{9419190750000000000000000000000000000}t^{36}$   
 38.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{38}}\right\} = \frac{1}{141287861250000000000000000000000000000}t^{37}$   
 39.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{39}}\right\} = \frac{1}{2119317918750000000000000000000000000000}t^{38}$   
 40.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{40}}\right\} = \frac{1}{31789768781250000000000000000000000000000}t^{39}$   
 41.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{41}}\right\} = \frac{1}{476836207368750000000000000000000000000000}t^{40}$   
 42.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{42}}\right\} = \frac{1}{7152543109375000000000000000000000000000000}t^{41}$   
 43.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{43}}\right\} = \frac{1}{107288146640625000000000000000000000000000000}t^{42}$   
 44.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{44}}\right\} = \frac{1}{1609322199609375000000000000000000000000000000}t^{43}$   
 45.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{45}}\right\} = \frac{1}{24139832994140625000000000000000000000000000000}t^{44}$   
 46.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{46}}\right\} = \frac{1}{362097494912109375000000000000000000000000000000}t^{45}$   
 47.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{47}}\right\} = \frac{1}{5431462423681640625000000000000000000000000000000}t^{46}$   
 48.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{48}}\right\} = \frac{1}{81471936355224609375000000000000000000000000000000}t^{47}$   
 49.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{49}}\right\} = \frac{1}{1222079045328369062500000000000000000000000000000000}t^{48}$   
 50.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{50}}\right\} = \frac{1}{18331185679925535937500000000000000000000000000000000}t^{49}$   
 51.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{51}}\right\} = \frac{1}{274967785198883031250000000000000000000000000000000000}t^{50}$   
 52.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{52}}\right\} = \frac{1}{4124516777983245468750000000000000000000000000000000000}t^{51}$   
 53.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{53}}\right\} = \frac{1}{61867751369748682031250000000000000000000000000000000000}t^{52}$   
 54.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{54}}\right\} = \frac{1}{928016270546230230468750000000000000000000000000000000000}t^{53}$   
 55.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{55}}\right\} = \frac{1}{13920244058193453453125000000000000000000000000000000000000}t^{54}$   
 56.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{56}}\right\} = \frac{1}{208803660862901800781250000000000000000000000000000000000000}t^{55}$   
 57.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{57}}\right\} = \frac{1}{3132054912943527011718750000000000000000000000000000000000000}t^{56}$   
 58.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{58}}\right\} = \frac{1}{46980823693152905175781250000000000000000000000000000000000000}t^{57}$   
 59.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{59}}\right\} = \frac{1}{7047123553972935776312500}t^{58}$   
 60.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{60}}\right\} = \frac{1}{105706853309594036645312500}t^{59}$   
 61.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{61}}\right\} = \frac{1}{15856027996439105496875000}t^{60}$   
 62.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{62}}\right\} = \frac{1}{237840419946586582453125000}t^{61}$   
 63.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{63}}\right\} = \frac{1}{356760629919880073687500}t^{62}$   
 64.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{64}}\right\} = \frac{1}{5351409438798161116875000}t^{63}$   
 65.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{65}}\right\} = \frac{1}{8027114158197241625000}t^{64}$   
 66.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{66}}\right\} = \frac{1}{12040671237295862500}t^{65}$   
 67.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{67}}\right\} = \frac{1}{180610068559437937500}t^{66}$   
 68.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{68}}\right\} = \frac{1}{2709151028391569062500}t^{67}$   
 69.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{69}}\right\} = \frac{1}{40637265425873535937500}t^{68}$   
 70.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{70}}\right\} = \frac{1}{6095589813881029375000}t^{69}$   
 71.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{71}}\right\} = \frac{1}{91433847208215437500}t^{70}$   
 72.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{72}}\right\} = \frac{1}{13715077081232315625000}t^{71}$   
 73.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{73}}\right\} = \frac{1}{205726156218484687500}t^{72}$   
 74.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{74}}\right\} = \frac{1}{30858923432772703125000}t^{73}$   
 75.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{75}}\right\} = \frac{1}{462883851491590500}t^{74}$   
 76.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{76}}\right\} = \frac{1}{69432577723738575000}t^{75}$   
 77.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{77}}\right\} = \frac{1}{104148866585607875000}t^{76}$   
 78.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{78}}\right\} = \frac{1}{156223300}t^{77}$   
 79.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{79}}\right\} = \frac{1}{2343349500}t^{78}$   
 80.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{80}}\right\} = \frac{1}{35150242500}t^{79}$   
 81.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{81}}\right\} = \frac{1}{527253637500}t^{80}$   
 82.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{82}}\right\} = \frac{1}{7908804562500}t^{81}$   
 83.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{83}}\right\} = \frac{1}{118632068437500}t^{82}$   
 84.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{84}}\right\} = \frac{1}{1779481026562500}t^{83}$   
 85.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{85}}\right\} = \frac{1}{26692215398437500}t^{84}$   
 86.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{86}}\right\} = \frac{1}{398373230976562500}t^{85}$   
 87.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{87}}\right\} = \frac{1}{5975488454648437500}t^{86}$   
 88.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{88}}\right\} = \frac{1}{8863228281972500}t^{87}$   
 89.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{89}}\right\} = \frac{1}{132948424229587500}t^{88}$   
 90.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{90}}\right\} = \frac{1}{20042263634437500}t^{89}$   
 91.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{91}}\right\} = \frac{1}{299633954516562500}t^{90}$   
 92.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{92}}\right\} = \frac{1}{4494518317748437500}t^{91}$   
 93.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{93}}\right\} = \frac{1}{6741777476622500}t^{92}$   
 94.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{94}}\right\} = \frac{1}{1011266621493375000}t^{93}$   
 95.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{95}}\right\} = \frac{1}{15169009322400625000}t^{94}$   
 96.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{96}}\right\} = \frac{1}{22753513983600937500}t^{95}$   
 97.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{97}}\right\} = \frac{1}{341302709754014062500}t^{96}$   
 98.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{98}}\right\} = \frac{1}{5119540646310210937500}t^{97}$   
 99.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{99}}\right\} = \frac{1}{767931096946531687500}t^{98}$   
 100.  $\text{Laplace}^{-1}\left\{\frac{1}{s^{100}}\right\} = \frac{1}{11518966454197984375000}t^{99}$

2. The input-output relation of a system  $\mathcal{S}$  is

$$y(t) = \int_{-\infty}^{\infty} \sin(t - \tau) u(t - 2\tau) x(\tau) d\tau.$$

→ This is not a convolutional integral!

(a) (2 points) Is the system time-invariant or time-varying?

(b) (3 points) What is the impulse response of the system?

(c) (5 points) Compute the output,  $y(t)$ , when the input is  $x(t) = u(t) - u(t - 1)$

Not the reason!!!

2a)  $h(t; \tau) = \sin(t - \tau) u(t - 2\tau)$  convolution integral

Time-varying Not a function of only  $t - \tau$  2

2b)  $T[\delta(t - \sigma)] = \int_{-\infty}^{\infty} \sin(t - \tau) u(t - 2\tau) \delta(\tau - \sigma) d\tau$

$h(t; \sigma) = \sin(t - \sigma) u(t - 2\sigma)$

$h(t; \tau) = \sin(t - \tau) u(t - 2\tau)$  3

2c)  $y(t) = T[x(t)] = T[u(t) - u(t - 1)]$

$= \int_{-\infty}^{\infty} \sin(t - \tau) u(t - 2\tau) (u(\tau) - u(\tau - 1)) d\tau$

$= \int_{-\infty}^{\infty} \sin(t - \tau) u(t - 2\tau) u(\tau) d\tau$

$- \int_{-\infty}^{\infty} \sin(t - \tau) u(t - 2\tau) u(\tau - 1) d\tau$

$= \int_0^{\infty} \sin(t - \tau) u(2(\frac{t}{2} - \tau)) d\tau - \int_1^{\infty} \sin(t - \tau) u(2(\frac{t}{2} - \tau)) d\tau$

$= \int_0^{t/2} \sin(t - \tau) d\tau - \int_1^{t/2} \sin(t - \tau) d\tau$  ✓

When  $t < 0$

$$y(t) = 0 \quad \checkmark$$

When  $0 < t < 2$

$$y(t) = \int_0^{t/2} \sin(t-\tau) d\tau$$

$$= \cos(t-\tau) \Big|_0^{t/2}$$

$$= \cos(t - t/2) - \cos(t)$$

$$= \cos(t/2) - \cos(t) \quad \checkmark$$

When  $t > 2$

$$y(t) = \int_0^{t/2} \sin(t-\tau) d\tau - \int_1^{t/2} \sin(t-\tau) d\tau$$

$$= \cos(t/2) - \cos(t) - \left( \cos(t-\tau) \Big|_1^{t/2} \right)$$

$$= \cos(t/2) - \cos(t) - (\cos(t/2) - \cos(t-1))$$

$$= \cos(t-1) - \cos(t) \quad \checkmark$$

5

$$y(t) = [\cos(t/2) - \cos(t)] u(t) u(2-t)$$

$$+ [\cos(t-1) - \cos(t)] u(t-2)$$

- LTIC
3. Let  $\mathcal{S}$  be a linear, time-invariant, and causal system. We know that the output corresponding to  $x(t) = (t-3)u(t-3)$  is

$$y(t) = 4e^{-(t-3)}u(t-3) - (t-4)u(t-4).$$

- (a) (3 points) What is the impulse response of  $\mathcal{S}$ ? (Hint: consider that  $\frac{d}{dt}tu(t) = u(t)$ .)  
 (b) (5 points) Compute the output of  $\mathcal{S}$  when  $x(t) = (1-t)u(t)$ .

3a)  $x(t) \rightarrow y(t) \quad h(t;\tau) = ?$

$$x(t+3) = tu(t)$$

$$\frac{d}{dt}x(t+3) = u(t)$$

$$T[u(t)] = T\left[\frac{d}{dt}x(t+3)\right] \\ = \frac{d}{dt}T[x(t+3)]$$

$$T[x(t+3)] = y(t+3)$$

$$= 4e^{-t}u(t) - (t-1)u(t-1)$$

$$\frac{d}{dt}y(t+3) = g(t)$$

$$g(t) = 4e^{-t}\delta(t) - 4e^{-t}u(t) - (u(t-1) + (t-1)\delta(t-1))$$

$$= 4\delta(t) - 4e^{-t}u(t) - u(t-1) - (0)\delta(t-1)$$

$$= 4\delta(t) - 4e^{-t}u(t) - u(t-1) \quad \text{continued on back}$$

3b)  $T[x(t)] = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

$$= \int_{-\infty}^{\infty} (4\delta(\tau) - 4e^{-\tau}u(\tau) - u(\tau-1))(1-(t-\tau))u(t-\tau)d\tau$$

$$= \int_{-\infty}^t (4\delta(\tau) - 4e^{-\tau}u(\tau) - u(\tau-1))(1-t+\tau)d\tau$$

$$= \underbrace{\int_{-\infty}^t 4\delta(\tau)(1-t+\tau)d\tau}_{I_1} - \underbrace{4\int_{-\infty}^t e^{-\tau}u(\tau)(1-t+\tau)d\tau}_{I_2} - \underbrace{\int_{-\infty}^t u(\tau-1)(1-t+\tau)d\tau}_{I_3}$$

$$3a) h(t) = \frac{d}{dt} g(t)$$

$$= 4\delta'(t) - 4(e^{-t}\delta(t) - e^{-t}u(t)) - \delta(t-1)$$

$$\boxed{= 4\delta'(t) - 4\delta(t) + 4e^{-t}u(t) - \delta(t-1)}$$



$$I_1 = \int_{-\infty}^{\infty} 4\delta(\tau)(1-t+\tau)u(t-\tau)d\tau$$

$$= 4(1-t)u(t)$$

$$I_2 = 4 \int_{-\infty}^t e^{-\tau} u(\tau)(1-t+\tau) d\tau$$

$$= 4 \int_0^t e^{-\tau} (1-t+\tau) d\tau \quad \begin{array}{l} u=1-t+\tau \quad dv=e^{-\tau} d\tau \\ du=d\tau \quad v=-e^{-\tau} \end{array}$$

$$= 4 \left( -e^{-\tau}(1-t+\tau) \Big|_0^t + \int_0^t e^{-\tau} d\tau \right)$$

$$= 4 \left( -e^{-t}(1-t) + (1-t) + [-e^{-\tau}]_0^t \right)$$

$$= 4(-e^{-t} + 1 - t - e^{-t} + 1)$$

$$= 4(-2e^{-t} + 2 - t) = -8e^{-t} + 4 - 4t$$

$$I_3 = \int_{-\infty}^t u(\tau-1)(1-t+\tau) d\tau$$

$$= \int_1^t (1-t+\tau) d\tau$$

$$= \left[ \tau + t\tau + \frac{\tau^2}{2} \right]_1^t = t + t^2 + \frac{t^2}{2} - \left( 1 + t + \frac{1}{2} \right)$$

$$= t^2 + \frac{t^2}{2} - \frac{3}{2} = \frac{3}{2}t^2 - \frac{3}{2}$$

$$y(t) = I_1 - I_2 - I_3$$

$$= 4(1-t)u(t) + 8e^{-t} - 4 + 4t - \frac{3}{2}t^2 + \frac{3}{2}$$

$$= 4u(t) - 4tu(t) + 8e^{-t} + 4t - \frac{3}{2}t^2 - \frac{1}{2}$$

$$3b) \quad x(t) = u(t) - tu(t)$$

$$\mathcal{T}[x(t)] = \mathcal{T}[u(t)] - \mathcal{T}[tu(t)]$$

$$= g(t) - \mathcal{T}[x(t+3)]$$

$$= g(t) - y(t+3)$$

$$= 4\delta'(t) - 4\delta(t) + 4e^{-t}u(t) - \delta(t-1)$$

$$- 4e^{-t}u(t) + (t-1)u(t-1)$$

$$\boxed{= 4\delta'(t) - 4\delta(t) - \delta(t-1) + (t-1)u(t-1)}$$



4. A system  $\mathcal{S}_1$  is described by

$$\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

- (a) (3 points) Write down the impulse response function  $h(t; \tau)$  of  $\mathcal{S}_1$ .  
 (b) (3 points) System  $\mathcal{S}_1$  is now cascaded with a second linear, time-invariant system,  $\mathcal{S}_2$  whose unit step response,  $g_2(t)$  is given by

LTI

$$g_2(t) = t e^{-t} u(t).$$

Compute the impulse response,  $h_{1,2}(t; \tau)$ , of the cascaded combination.

4a)  $M = e^{\int 2 dt} = e^{2t}$

$$\frac{d}{dt} [e^{2t} y(t)] = e^{2t} \left( \frac{dx}{dt} - 2x(t) \right)$$

$$\int_0^t \frac{d}{d\tau} [e^{2\tau} y(\tau)] d\tau = \int_0^t e^{2\tau} \left( \frac{dx}{d\tau} - 2x(\tau) \right) d\tau$$

$$e^{2t} y(t) = \int_0^t e^{2\tau} \left( \frac{dx}{d\tau} - 2x(\tau) \right) d\tau$$

$$y(t) = \frac{\int_0^t e^{2\tau} \left( \frac{dx}{d\tau} - 2x(\tau) \right) d\tau}{e^{2t}} \quad 2$$

$$\mathcal{T}[\delta(t-\sigma)] = \frac{\int_0^t e^{2\tau} \left( \frac{dx}{d\tau} - 2\delta(\tau-\sigma) \right) d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau} \frac{dx}{d\tau} d\tau - 2 \int_0^t e^{2\tau} \delta(\tau-\sigma) d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau} \frac{dx}{d\tau} d\tau - 2 \int_{-\infty}^{\infty} e^{2\tau} u(\tau) u(t-\tau) \delta(\tau-\sigma) d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau} \frac{dx}{d\tau} d\tau - 2 e^{2\sigma} u(\sigma) u(t-\sigma)}{e^{2t}}$$

$$h(t; \tau) = \frac{\int_0^t e^{2\sigma} \frac{dx}{d\sigma} d\sigma - 2 e^{2\tau} u(\tau) u(t-\tau)}{e^{2t}} \quad \text{LTI?}$$

$$S_2: T[u(t)] = t e^{-t} u(t) = g(t)$$

$$\frac{d}{dt} g(t) = h_2(t)$$

$$\begin{aligned} h_2(t) &= t' e^{-t} u(t) + t (e^{-t} u(t))' \\ &= e^{-t} u(t) + t (e^{-t} \delta(t) - e^{-t} u(t)) \\ &= e^{-t} u(t) + t (\delta(t) - e^{-t} u(t)) \\ &= e^{-t} u(t) + t \delta(t) - t e^{-t} u(t) \\ &= e^{-t} u(t) - t e^{-t} u(t) \end{aligned}$$

$$\begin{aligned} h_{1,2}(t; \tau) &= \int_{-\infty}^{\infty} h_2(t; \sigma) h_1(\sigma; \tau) d\sigma = \underline{e^{-t} u(t) (1-t)} \\ &= \int_{-\infty}^{\infty} h_2(t-\sigma) h_1(\sigma; \tau) d\sigma \end{aligned}$$

$$= \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(t-\sigma) (1-t+\sigma) \frac{\int_0^{\sigma} e^{2\alpha} \frac{d\alpha}{d\alpha} d\alpha - 2e^{2t} u(t) u(\sigma-t)}{e^{2\sigma}} d\sigma$$

$$= e^{-t} \int_{-\infty}^t e^{-\sigma} (1+t+\sigma) \left( \int_0^{\sigma} e^{2\alpha} \frac{d\alpha}{d\alpha} d\alpha - 2e^{2t} u(t) u(\sigma-t) \right) d\sigma$$

$$= e^{-t} \left( \int_{-\infty}^t e^{-\sigma} (1+t+\sigma) \left( \int_0^{\sigma} e^{2\alpha} \frac{d\alpha}{d\alpha} d\alpha \right) d\sigma \right.$$

$$\left. - \int_{-\infty}^t e^{-\sigma} (1+t+\sigma) 2e^{2t} u(t) u(\sigma-t) d\sigma \right)$$

$$\int_0^t 2e^{-\sigma} e^{2t} (1+t+\sigma) u(\sigma-t) d\sigma$$

Product to Sum	Sum to Product
$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$	$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$
$2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$	$\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$
$2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$	$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$
$2 \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta)$	$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$
Sum/Difference	Pythagorean Identity
$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$	$\sin^2(\alpha) + \cos^2(\alpha) = 1$
$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$	
$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \tan(\beta)}$	
Even/Odd	Periodic Identities
$\sin(-\alpha) = -\sin(\alpha)$	$\sin(\alpha + 2\pi n) = \sin(\alpha)$
$\cos(-\alpha) = \cos(\alpha)$	$\cos(\alpha + 2\pi n) = \cos(\alpha)$
$\tan(-\alpha) = -\tan(\alpha)$	$\tan(\alpha + \pi n) = \tan(\alpha)$
Double-Angle Identities	Half-Angle Identities
$\sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$	$\sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 - \cos(\alpha)}{2}}$
$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$	$\cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 + \cos(\alpha)}{2}}$
$\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$	$\tan\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 - \cos(\alpha)}{1 + \cos(\alpha)}}$
Laws of Sines, Cosines, and Tangents	Mollweide's Formula
$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$	$\frac{a + b}{c} = \frac{\cos\left(\frac{\alpha - \beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$
$a^2 = b^2 + c^2 - 2bc \cos(\alpha)$	
$\frac{a - b}{a + b} = \frac{\tan\left(\frac{\alpha - \beta}{2}\right)}{\tan\left(\frac{\alpha + \beta}{2}\right)}$	