UCLA — Electrical Engineering Dept.
EE102: Systems and Signals — Midterm Exam
Wednesday, October 28, 2015

This exam has 4 questions, for a total of 33 points.

Closed book. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
Please, write your name and ID on the top of each loose sheet!

Name and ID: [redacted]

Name of person on your left: [redacted]

Name of person on your right: [redacted]

Question	Points	Score
1	9	8
2	10	10
3	8	7.5
4	6	3,5
Total:	33	29

Multiple-choice questions – Check all the answers that apply.

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

$K_1 x_1(t) + k_2 x_2(t)$

$\sin(k_1 x_1(t) + k_2 x_2(t))$

Non linear

What are its properties?

- [ ] Linear
- [x] Time-invariant
- [ ] Causal

$z(t) = x(t - t_0)$

$T[z(t)] = \sin(z(t+1))$

$= \sin(x(t+1-t_0))$

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} e^{t} e^{-\tau} x(\tau) u(t-\tau)\, d\tau.$$

$[\delta(t-\sigma)]$

$= \int_{-\infty}^{\infty} e^{t} e^{-\tau} \delta(\tau-\sigma)$

$\delta(t-\sigma)$

What are its properties?

- [x] Linear
- [ ] Time-invariant
- [x] Causal

$y(t) = \int_{-\infty}^{\infty} e$

$h(t;\tau) = e^{t} e^{-\tau} u(t-\tau)$

$t - \tau < 0$

$e^{t-\tau}$

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

What is the condition on a for the system to be causal?

- [ ] $a < 1$
- [x] $a = 1$
- [ ] $a > 1$

$x(0.5t)$ $t = -1$ $t = -0.5$

$x(-0.5)$ $x(-0.25)$

$x(t)$

$T[\delta(t-\sigma)] = \int_{-\infty}^{\infty} e^{t} e^{-\tau} \delta(\tau-\sigma) u(t-\tau)\, d\tau$

$= e^{t} e^{-\sigma} u(t-\sigma)$

$y(t-t_0) = \int_{-\infty}^{\infty} e^{t-t_0} e^{-\tau} x(\tau) u(t-t_0-\tau)\, d\tau$

$z(t) = x(t-t_0)$

$T[z(t)] = \int_{-\infty}^{\infty} e^{t} e^{-\tau} z(\tau) u(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} e^{t} e^{-\tau} x(\tau - t_0) u(t-\tau)\, d\tau$

2. The input-output relation of a system $\mathcal{S}$ is

$$y(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\mathrm{u}(t-2\tau)x(\tau)\,\mathrm{d}\tau.$$

→ This is not a convolutional integral!

(a) (2 points) Is the system time-invariant or time-varying?

(b) (3 points) What is the impulse response of the system?

(c) (5 points) Compute the output, $y(t)$, when the input is $x(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

Not the reason!!!

2a) $h(t;\tau) = \sin(t-\tau)\,u(t-2\tau)$

convolution integral

Time-varying   Not a function of only $t-\tau$

$\boxed{2}$

2b) $T[\delta(t-\sigma)] = \int_{-\infty}^{\infty} \sin(t-\tau)\,u(t-2\tau)\,\delta(\tau-\sigma)\,d\tau$

$h(t;\sigma) = \sin(t-\sigma)\,u(t-2\sigma)$

$h(t;\tau) = \sin(t-\tau)\,u(t-2\tau)$

$\boxed{3}$

2c) $y(t) = T[x(t)] = T[u(t) - u(t-1)]$

$= \int_{-\infty}^{\infty} \sin(t-\tau)\,u(t-2\tau)(u(\tau) - u(\tau-1))\,d\tau$

$= \int_{-\infty}^{\infty} \sin(t-\tau)\,u(t-2\tau)\,u(\tau)\,d\tau$

$- \int_{-\infty}^{\infty} \sin(t-\tau)\,u(t-2\tau)\,u(\tau-1)\,d\tau$

$= \int_{0}^{\infty} \sin(t-\tau)\,u\left(2\left(\frac{t}{2}-\tau\right)\right)d\tau - \int_{1}^{\infty} \sin(t-\tau)\,u\left(2\left(\frac{t}{2}-\tau\right)\right)d\tau$

$= \int_{0}^{t/2} \sin(t-\tau)\,d\tau - \int_{1}^{t/2} \sin(t-\tau)\,d\tau$ ✓

When $t<0$

$y(t) = 0$ ✓

when $0 < t < 2$

$$y(t) = \int_0^{t/2} \sin(t-\tau)\,d\tau$$

$$= \cos(t-\tau)\Big|_0^{t/2}$$

$$= \cos(t - t/2) - \cos(t)$$

$$= \cos(t/2) - \cos(t)$$ ✓

when $t > 2$

$$y(t) = \int_0^{t/2} \sin(t-\tau)\,d\tau - \int_1^{t/2} \sin(t-\tau)\,d\tau$$

$$= \cos(t/2) - \cos(t) - \left(\cos(t-\tau)\Big|_1^{t/2}\right)$$

$$= \cos(t/2) - \cos(t) - \left(\cos(t/2) - \cos(t-1)\right)$$

$\boxed{5}$

$$= \cos(t-1) - \cos(t)$$ ✓

$$y(t) = [\cos(t/2) - \cos(t)]\,u(t)\,u(2-t)$$

$$+ [\cos(t-1) - \cos(t)]\,u(t-2)$$

LTIC

3. Let $\mathcal{S}$ be a linear, time-invariant, and causal system. We know that the output corresponding to $x(t) = (t-3)\mathrm{u}(t-3)$ is

$$y(t) = 4\mathrm{e}^{-(t-3)}\mathrm{u}(t-3) - (t-4)\mathrm{u}(t-4).$$

(a) (3 points) What is the impulse response of $\mathcal{S}$? (Hint: consider that $\frac{\mathrm{d}}{\mathrm{d}t}t\mathrm{u}(t) = \mathrm{u}(t)$.)

(b) (5 points) Compute the output of $\mathcal{S}$ when $x(t) = (1-t)\mathrm{u}(t)$.

3a) $x(t) \rightarrow y(t) \quad h(t;\tau) = ?$

$x(t+3) = t\,u(t)$

$\frac{d}{dt} x(t+3) = u(t) \qquad T[u(t)] = T[\frac{d}{dt} x(t+3)]$

$\qquad = \frac{d}{dt} T[x(t+3)]$

$T[x(t+3)] = y(t+3)$

$= 4e^{-t}u(t) - (t-1)u(t-1)$

$\frac{d}{dt} y(t+3) = g(t)$

$g(t) = 4e^{-t}\delta(t) - 4e^{-t}u(t) - \left(u(t-1) + (t-1)\delta(t-1)\right)$

$= 4\delta(t) - 4e^{-t}u(t) - u(t-1) - (0)\delta(t-1)$

$= 4\delta(t) - 4e^{-t}u(t) - u(t-1)$ continued on back

3b) $T[x(t)] = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} (4\delta(\tau) - 4e^{-\tau}u(\tau) - u(\tau-1))(1-(t-\tau))u(t-\tau)\,d\tau$

$= \int_{-\infty}^{t} (4\delta(\tau) - 4e^{-\tau}u(\tau) - u(\tau-1))(1-t+\tau)\,d\tau$

$= \underbrace{\int_{-\infty}^{t} 4\delta(\tau)(1-t+\tau)\,d\tau}_{I_1} - \underbrace{4\int_{-\infty}^{t} e^{-\tau}u(\tau)(1-t+\tau)\,d\tau}_{I_2} - \underbrace{\int_{-\infty}^{t} u(\tau-1)(1-t+\tau)\,d\tau}_{I_3}$

3a) $h(t) = \frac{d}{dt} g(t)$

$= 4\delta'(t) - 4(e^{-t}\delta(t) - e^{-t}u(t)) - \delta(t-1)$

$\boxed{= 4\delta'(t) - 4\delta(t) + 4e^{-t}u(t) - \delta(t-1)}$ ✓

$I_1 = \int_{-\infty}^{\infty} 4\delta(\tau)(1-t+\tau)u(t-\tau)\,d\tau$

$= 4(1-t)u(t)$

$I_2 = 4\int_{-\infty}^{t} e^{-\tau}u(\tau)(1-t+\tau)\,d\tau$

$= 4\int_0^t e^{-\tau}(1-t+\tau)\,d\tau \qquad u=(1-t+\tau) \quad dv = e^{-\tau}d\tau$

$du = d\tau \qquad v = -e^{-\tau}$

$= 4\left(-e^{-\tau}(1-t+\tau)\Big|_0^t + \int_0^t e^{-\tau}d\tau\right)$

$= 4\left(-e^{-t}(1) + (1-t) + [-e^{-\tau}]_0^t\right)$

$= 4(-e^{-t} + 1 - t - e^{-t} + 1)$

$= 4(-2e^{-t} + 2 - t) = -8e^{-t} + 4 - 4t$

$I_3 = \int_{-\infty}^{t} u(\tau-1)(1-t+\tau)\,d\tau$

$= \int_1^t 1-t+\tau\,d\tau$

$= \left[\tau + t\tau + \frac{t^2}{2}\right]_1^t = t + t^2 + \frac{t^2}{2} - \left(1 + t + \frac{1}{2}\right)$

$= t^2 + \frac{t^2}{2} - \frac{3}{2} = \frac{3}{2}t^2 - \frac{3}{2}$

$y(t) = I_1 - I_2 - I_3$

$= 4(1-t)u(t) + 8e^{-t} - 4 + 4t - \frac{3}{2}t^2 + \frac{3}{2}$

$= 4u(t) - 4tu(t) + 8e^{-t} + 4t - \frac{3}{2}t^2 - \frac{1}{2}$

3b) $x(t) = u(t) - tu(t)$

$T[x(t)] = T[u(t)] - T[tu(t)]$

$= g(t) - T[x(t+3)]$

-0.5

$= g(t) - y(t+3)$

$= 4\delta'(t) - 4\delta(t) + 4e^{-t}u(t) - \delta(t-1)$

$- 4e^{-t}u(t) + (t-1)u(t-1)$

$= 4\delta'(t) - 4\delta(t) - \delta(t-1) + (t-1)u(t-1)$

4. A system $\mathcal{S}_1$ is described by

$$\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

(a) (3 points) Write down the impulse response function $h(t;\tau)$ of $\mathcal{S}_1$.

(b) (3 points) System $\mathcal{S}_1$ is now cascaded with a second linear, time-invariant system, $\mathcal{S}_2$ whose unit step response, $g_2(t)$ is given by

LTI

$$g_2(t) = t\,e^{-t}u(t).$$

Compute the impulse response, $h_{1,2}(t;\tau)$, of the cascaded combination.

4a) $M = e^{\int 2\,dt} = e^{2t}$

$$\frac{d}{dt}\left[e^{2t}y(t)\right] = e^{2t}\left(\frac{dx}{dt} - 2x(t)\right)$$

$$\int_0^t \frac{d}{d\tau}\left[e^{2\tau}y(\tau)\right]d\tau = \int_0^t e^{2\tau}\left(\frac{dx}{d\tau} - 2x(\tau)\right)d\tau$$

$$e^{2t}y(t) = \int_0^t e^{2\tau}\left(\frac{dx}{d\tau} - 2x(\tau)\right)d\tau$$

$$y(t) = \frac{\int_0^t e^{2\tau}\left(\frac{dx}{d\tau} - 2x(\tau)\right)d\tau}{e^{2t}}$$

-2

$$T[\delta(t-\sigma)] = \frac{\int_0^t e^{2\tau}\left(\frac{dx}{d\tau}{}^{?} - 2\delta(\tau-\sigma)\right)d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau}\frac{dx}{d\tau}d\tau - 2\int_0^t e^{2\tau}\delta(\tau-\sigma)\,d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau}\frac{dx}{d\tau}\,d\tau - 2\int_{-\infty}^{\infty} e^{2\tau}u(\tau)\,u(t-\tau)\,\delta(\tau-\sigma)\,d\tau}{e^{2t}}$$

$$= \frac{\int_0^t e^{2\tau}\frac{dx}{d\tau}d\tau - 2e^{2\sigma}u(\sigma)\,u(t-\sigma)}{e^{2t}}$$

$$h(t;\tau) = \frac{\int_0^t e^{2\sigma}\frac{dx}{d\sigma}d\sigma - 2e^{2\tau}u(\tau)\,u(t-\tau)}{e^{2t}}$$

LTI?

$S_2: T[u(t)] = te^{-t}u(t) = g(t)$

$\frac{d}{dt} g(t) = h_2(t)$

$$h_2(t) = t' e^{-t}u(t) + t(e^{-t}u(t))'$$
$$= e^{-t}u(t) + t(e^{-t}\delta(t) - e^{-t}u(t))$$
$$= e^{-t}u(t) + t(\delta(t) - e^{-t}u(t))$$
$$= e^{-t}u(t) + t\delta(t) - te^{-t}u(t)$$
$$= e^{-t}u(t) - te^{-t}u(t) \longrightarrow = \underline{e^{-t}u(t)(1-t)}$$

$$h_{1,2}(t;\tau) = \int_{-\infty}^{\infty} h_2(t;\sigma)\, h_1(\sigma;\tau)\, d\sigma$$
$$= \int_{-\infty}^{\infty} h_2(t-\sigma)\, h_1(\sigma;\tau)\, d\sigma$$

$$= \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(t-\sigma)(1-t+\sigma) \frac{\int_0^{\sigma} e^{2\alpha}\frac{dx}{d\alpha}\, d\alpha - 2e^{2t}u(\cancel{t})u(\sigma-t)}{e^{2\sigma}} d\sigma$$

-0.5

$$= e^{-t}\int_{-\infty}^{t} e^{-\sigma}(1+t+\sigma)\left(\int_0^{\sigma} e^{2\alpha}\frac{dx}{d\alpha}\, d\alpha - 2e^{2t}u(\tau)u(\sigma-\tau)\right) d\sigma$$

$$= e^{-t}\Bigg(\int_{-\infty}^{t} e^{-\sigma}(1+t+\sigma)\left(\int_0^{\sigma} e^{2\alpha}\frac{dx}{d\alpha}\, d\alpha\right) d\sigma$$
$$- \underbrace{\int_{-\infty}^{t} e^{-\sigma}(1+t+\sigma)2e^{2t}u(\tau)u(\sigma-\tau)\, d\sigma}\Bigg)$$

$$\int_0^{t} 2e^{-\sigma}e^{2t}(1+t+\sigma)u(\sigma-\tau)\, d\sigma$$

Product to Sum	Sum to Product
$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ $2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta)$ $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ $2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$	$\sin(\alpha)+\sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin(\alpha)-\sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
Sum/Difference	**Pythagorean Identity**
$\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$ $\tan(\alpha\pm\beta) = \dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$	$\sin^2(\alpha)+\cos^2(\alpha) = 1$
Even/Odd	**Periodic Identities**
$\sin(-\alpha) = -\sin(\alpha)$ $\cos(-\alpha) = \cos(\alpha)$ $\tan(-\alpha) = -\tan(\alpha)$	$\sin(\alpha+2\pi n) = \sin(\alpha)$ $\cos(\alpha+2\pi n) = \cos(\alpha)$ $\tan(\alpha+\pi n) = \tan(\alpha)$
Double-Angle Identities	**Half-Angle Identities**
$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ $\cos(2\alpha) = \cos^2(\alpha)-\sin^2(\alpha)$ $\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1-\tan^2(\alpha)}$	$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$ $\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos(\alpha)}{2}}$ $\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}}$
Laws of Sines, Cosines, and Tangents	**Mollweide's Formula**
$\dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$ $a^2 = b^2+c^2-2bc\cos(\alpha)$ $\dfrac{a-b}{a+b} = \dfrac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$	$\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$