This exam has 4 questions, for a total of 33 points.

Closed book. One two-sided cheat-sheet allowed.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
**Please, write your name and ID on the top of each loose sheet!**

Name and ID: Max Chern

Name of person on your left: None

Name of person on your right: Vanessa Chiang

| Question | Points | Score |
|---|---|---|
| 1 | 9 | 6 |
| 2 | 10 | 7 |
| 3 | 8 | 8 |
| 4 | 6 | 3.5 |
| Total: | 33 | 24.5 |

Max Chern

**Multiple-choice questions – Check all the answers that apply.**

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

$k_1x_1(t)+k_2x_2(t)$

$y(t) = \sin(k_1x_1(t+1) + k_2x_2(t+1))$

What are its properties?

- ☐ Linear
- ☑ Time-invariant
- ☐ Causal

$y(t-\tau) = \sin(x(t-\tau+1))$

$y'(t) = \sin(x(t-\tau+1))$

$y(t)$ depends on $x(t+1)$

NC

✓

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} e^{t}e^{-\tau}x(\tau)u(t-\tau)\,d\tau.$$

$\tau = \theta + \sigma$

$\tau - \sigma = \theta$

$y(t) = \int_{-\infty}^{\infty} e^{t}e^{-(\theta+\sigma)} x(\theta)u(t-\theta-\sigma)\,d\sigma$

$y(t) = \int_{-\infty}^{\infty} e^{t}e^{-\tau}x(\tau-\sigma)u(t-\tau)\,d\tau$

What are its properties?

- ☑ Linear
- ☐ Time-invariant
- ☐ Causal — NC, bc integral goes to $\infty$

$y(t-\sigma) = \int_{-\infty}^{\infty} e^{t-\sigma}e^{-\tau}x(\tau)u(t-\sigma-\tau)\,d\tau$

-2

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

What is the condition on $a$ for the system to be causal?

- ☑ $a < 1$
- ☑ $a = 1$
- ☐ $a > 1$

-1

Max Chern

2. The input-output relation of a system $\mathcal{S}$ is

$$y(t) = \int_{-\infty}^{\infty} \underbrace{\sin(t-\tau)\mathrm{u}(t-2\tau)}_{h(t;\tau)}\underbrace{x(\tau)}_{x(\tau)}\, d\tau.$$

→ This is not a convolution integral. At first you cannot directly identy h, x separately. That is logically incorrect.

(a) (2 points) Is the system time-invariant or time-varying?

(b) (3 points) What is the impulse response of the system?

(c) (5 points) Compute the output, $y(t)$, when the input is $x(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

a. Time variant

because $h(t;\tau)$ has a $u(t-2\tau)$ term; not only terms that depend on $t-\tau$

OK 2

b. Linear System, so

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t;\tau)\, d\tau$$

3

$$\boxed{h(t;\tau) = \sin(t-\tau)\,u(t-2\tau)}$$

20

c. $y(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\, u(t-2\tau)\left[u(t) - u(t-1)\right] d\tau$

$= \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)u(t)\, d\tau - \int_{-\infty}^{\infty} \sin(t-\tau)\,u(t-2\tau)\,u(t-1)\, d\tau$

$= \underbrace{\int_0^{\frac{t}{2}} \sin(t-\tau)\, d\tau}_{x u(t)} - \underbrace{\int_1^{\frac{t}{2}} \sin(t-\tau)\, d\tau}_{x u(t-2)} = \int_0^1 \sin(t-\tau)\, d\tau = \cos(t-\tau)\Big|_0^1 = \boxed{\cos(t-1) - \cos(t)}$

No.

$\cos(t-\tau)$



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$t - 2\tau = 0$

$t = 2\tau$

$\tau = \frac{t}{2}$

Max Chen

LTIC

3. Let $\mathcal{S}$ be a linear, time-invariant, and causal system. We know that the output corresponding to $x(t) = (t-3)\mathrm{u}(t-3)$ is

$$y(t) = 4e^{-(t-3)}\mathrm{u}(t-3) - (t-4)\mathrm{u}(t-4).$$

(a) (3 points) What is the impulse response of $\mathcal{S}$? (Hint: consider that $\frac{\mathrm{d}}{\mathrm{d}t}t\mathrm{u}(t) = \mathrm{u}(t)$.)

(b) (5 points) Compute the output of $\mathcal{S}$ when $x(t) = (1-t)\mathrm{u}(t)$.

$u(t) + t\delta(t) = u(t)$

$u(t-1) + (t-1)\delta(t-1) = u(t-1)$

$T[x(t)] = y(t)$

$T[(t-3)u(t-3)] = 4e^{-(t-3)}u(t-3) - (t-4)u(t-4)$

TI, so ...

$T[t\,u(t)] = 4e^{-t}u(t) - (t-1)u(t-1)$ ✓

Derivative of both

$T[u(t)] = 4e^{-t}\overset{t=0}{\delta(t)} + (-4e^{-t}u(t)) - u(t-1)$

$= 4\delta(t) - 4e^{-t}u(t) - u(t-1)$

$h(t) = T[\delta(t)] = [4\delta(t)]' - [-4e^{-t}u(t) + 4e^{-t}\delta(t)] - \delta(t-1)$

$= 4\delta'(t) + 4e^{-t}u(t) - 4\delta(t) - \delta(t-1)$ ✓

~~Is $\frac{d}{dt}\delta(t) = 0$? I don't know it~~

Thanks lol !!!

b. ~~$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$~~

~~$= \int_{-\infty}^{\infty}(1-\tau)u(\tau)h(t-\tau)\,d\tau$~~

~~$= \int_{0}^{\infty}(1-\tau)h(t-\tau)\,d\tau$~~

~~$= 4\int_0^\infty (1-\tau)\delta'(\tau)\,d\tau + 4\int_0^\infty(1-\tau)e^{-\tau}\,d\tau - 4\int_0^\infty(1-\tau)\delta(\tau)\,d\tau$~~

~~$- \int_0^\infty (1-\tau)\delta(\tau-1)\,d\tau$~~

~~$= 4(1-\tau)\delta(\tau) - 4 +$~~

$u = 1-\tau \quad dv = e^{-\tau}d\tau$

$du = -1\,d\tau \quad v = -e^{-\tau}d\tau$

$u = 1-\tau \quad dv = \delta'(\tau)\,d\tau$

$du = -1\,d\tau \quad v = \delta(\tau)$

$= (1-\tau)\delta(\tau)\Big|_0^\infty - \int_0^\infty \delta(\tau)\,d\tau$

1?

$-(1-\tau)e^{-\tau}d\tau - \int e^{-\tau}d\tau$



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Max Chan

b. $h(t) = 4\delta'(t) + 4e^{-t}u(t) - 4\delta(t) - \delta(t-1)$

$x(t) = (1-t)u(t)$

$T[(1-t)(u(t))]$

$= T[u(t) - tu(t)]$

$= T[u(t)] - T[tu(t)]$

$= \left[4\delta(t) - 4e^{-t}u(t) - u(t-1)\right] - \left[4e^{-t}u(t) - (t-1)u(t-1)\right]$

$= 4\delta(t) - 8e^{-t}u(t) + (t-1)u(t-1) - u(t-1)$

$= 4\delta(t) - 8e^{-t}u(t) + (t-2)u(t-1)$

Max Chem

4. A system $S_1$ is described by

$$\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

(a) (3 points) Write down the impulse response function $h(t;\tau)$ of $S_1$.

(b) (3 points) System $S_1$ is now cascaded with a second linear, time-invariant system, $S_2$ whose unit step response, $g_2(t)$ is given by

$$g_2(t) = te^{-t}u(t).$$

Compute the impulse response, $h_{1,2}(t;\tau)$, of the cascaded combination.

a. $e^{2t}\frac{dy}{dt} + 2e^{2t}y(t) = e^{2t}\frac{dx}{dt} - 2e^{2t}x(t)$

$e^{2t}y(t)\Big|_0^t = \int_0^t e^{2t}\frac{dx}{dt}\,dt - \int_0^t 2e^{2\tau}x(t)\,d\tau$

$y(0)=0$

$e^{2t}y(t) = \int_0^t e^{2t}\,dx - \int_0^t 2e^{2\tau}x(t)\,d\tau$

$e^{2t}y(t) = e^{2t}x\Big|_0^t - \int_0^t 2e^{2\tau}x(\tau)\,d\tau$

$e^{2t}y(t) = te^{2t} - 2\int_0^t e^{2\tau}x(\tau)\,d\tau$

$-1$

$$y(t) = t - \frac{2\int_0^t e^{2\tau}x(\tau)\,d\tau}{e^{2t}}$$

$\sigma = \tau$

$$h(t;\tau) = t - \frac{2\int_0^t e^{2\sigma}\delta(\sigma-\tau)\,d\sigma}{e^{2t}}$$

$$= t - \frac{2\int_{-\infty}^{\infty} e^{2\sigma}\delta(\sigma-\tau)u(t-\sigma)\,d\sigma}{e^{2t}}$$

$$= t - \frac{2e^{2\tau}u(t-\tau)}{e^{2t}}, \quad t>0$$

Max Chern

b. $g_2(t) = te^{-t}u(t)$

$$\frac{d}{dt} g_2(t) = h_2(t) = \left[(t)\left[(e^{-t})\,u(t)\right]\right]'$$

$$= e^{-t}u(t) + (t)\left[-e^{-t}u(t) + e^{-t}\delta(t)\right]$$

$$= e^{-t}u(t) - e^{-t}\,t\,u(t) + e^{-t}t\,\delta(t)$$

$$= e^{-t}\left[u(t) - tu(t) + t\delta(t)\right] \quad t=0$$

$$= e^{-t}\left[u(t) - tu(t) + \underbrace{\delta(t)}_{=0}\right] \qquad -1/2$$

Input B y(t)

Convolution or Superposition?

$$h_{1,2}(t;\tau) = \int_{-\infty}^{\infty} h_1(t;\tau)\,h_2(t)\,d\tau$$

$$= \int_{-\infty}^{\infty} \tau e^{-\tau}u(t)\,d\tau + \int_{-\infty}^{\infty} \tau^2 e^{-\tau}u(\tau)\,d\tau + \cancel{\int_{-\infty}^{\infty} \tau e^{-\tau}\delta(\tau)\,d\tau} + \int_{-\infty}^{\infty} \frac{2e^{2\tau}u(\sigma-\tau)}{e^{2\sigma}}e^{-\sigma}u(\sigma)\,d\sigma$$

$$+ \int_{-\infty}^{\infty} \frac{2e^{2\tau}u(\sigma-\tau)}{e^{2\sigma}}e^{-\sigma}\sigma u(\sigma)\,d\sigma$$

$$+ \int_{-\infty}^{\infty} \frac{2e^{2\tau}u(\sigma-\tau)}{e^{2\sigma}}\delta(\sigma)\,d\sigma$$

→ pretend all τs are σs

$$= \int_0^{\infty} \sigma e^{-\sigma}\,d\sigma + \int_0^{\infty} \sigma^2 e^{-\sigma}\,d\sigma + \int_{\tau}^{\infty} \frac{2e^{2\tau}}{e^{3\sigma}}\,d\sigma + \int_{\tau}^{\infty} \frac{2e^{2\tau}}{e^{3\sigma}}\,d\sigma + \int_{\tau}^{\infty} 2e^{2\tau}\delta(\sigma)\,d\sigma$$

$$= \cancel{-\sigma e^{-\sigma}\Big|_0^{\infty}} - \int_0^{\infty} e^{-\sigma}\,d\sigma - \sigma^2 e^{-\sigma}\Big|_0^{\infty} + \int_0^{\infty} 2\sigma e^{-\sigma}\,d\sigma$$

No time, I probably missed something that would have made this easier :(

$u = \sigma$ $\quad dv = e^{-\sigma}d\sigma$ $\quad u = \sigma^2$ $\quad dV = e^{-\sigma}d\sigma$ $\qquad -1$

$du = 1$ $\quad V = -e^{-\sigma}$ $\quad du = 2\sigma\,d\sigma$ $\quad V = -e^{-\sigma}$

| Product to Sum | Sum to Product |
|---|---|
| $2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$<br>$2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta)$<br>$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$<br>$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$ | $\sin(\alpha)+\sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$<br>$\sin(\alpha)-\sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$<br>$\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$<br>$\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ |
| **Sum/Difference** | **Pythagorean Identity** |
| $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$<br>$\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$<br>$\tan(\alpha\pm\beta) = \dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$ | $\sin^2(\alpha)+\cos^2(\alpha) = 1$ |
| **Even/Odd** | **Periodic Identities** |
| $\sin(-\alpha) = -\sin(\alpha)$<br>$\cos(-\alpha) = \cos(\alpha)$<br>$\tan(-\alpha) = -\tan(\alpha)$ | $\sin(\alpha+2\pi n) = \sin(\alpha)$<br>$\cos(\alpha+2\pi n) = \cos(\alpha)$<br>$\tan(\alpha+\pi n) = \tan(\alpha)$ |
| **Double-Angle Identities** | **Half-Angle Identities** |
| $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$<br>$\cos(2\alpha) = \cos^2(\alpha)-\sin^2(\alpha)$<br>$\tan(2\alpha) = \dfrac{2\tan(\alpha)}{1-\tan^2(\alpha)}$ | $\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$<br>$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos(\alpha)}{2}}$<br>$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{1+\cos(\alpha)}}$ |
| **Laws of Sines, Cosines, and Tangents** | **Mollweide's Formula** |
| $\dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$<br>$a^2 = b^2+c^2-2bc\cos(\alpha)$<br>$\dfrac{a-b}{a+b} = \dfrac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$ | $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$ |