Multiple-choice questions – **Check all the answers that apply.**

1. (a) (3 points) Consider the system described by the input-output relation

$$y(t) = \sin(x(t+1)).$$

$T[k_x x_1(t) + k_2 x_2(t)] = \sin(k_x x(t+1) + k_2 x(t+2))$

What are its properties?

☐ Linear

☒ Time-invariant

☐ Causal (Depends on $x(t+1)$)

(b) (3 points) Consider the system described by the input-output relation

$$y(t) = \int_{-\infty}^{\infty} e^{t} e^{-\tau} x(\tau) u(t-\tau)\, d\tau. = \int_{-\infty}^{t} e^{t-\tau} x(\tau)\, d\tau$$

What are its properties?

☒ Linear

☒ Time-invariant

☒ Causal

(c) (3 points) Consider the system described by the input-output relation

$$y(t) = x(at), \quad a > 0.$$

What is the condition on a for the system to be causal?

☐ $a < 1$ assuming $-\infty < t < \infty$, since if $a = 1/2$

☒ $a = 1$

☐ $a > 1$

& $t = -4$, then $y(-4) = x(-2)$

2. The input-output relation of a system $\mathcal{S}$ is

$$y(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\mathrm{u}(t-2\tau)x(\tau)\,\mathrm{d}\tau.$$

(a) (2 points) Is the system time-invariant or time-varying?

(b) (3 points) What is the impulse response of the system?

(c) (5 points) Compute the output, $y(t)$, when the input is $x(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$ $= u(t)u(1-t)$

a) let $z(t) = x(t-\sigma)$

$$T[z(t)] = \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)z(\tau)\,d\tau = \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)x(\tau-\sigma)\,d\tau$$

$$y(t-\sigma) = \int_{-\infty}^{\infty} \sin(t-\sigma-\tau)u(t-\sigma-2\tau)x(\tau)\,d\tau$$

OK but it is not so obvious

Time varying system $\boxed{2}$

b) Change the system's τ to sigma because it's confusing:

$$y(t) = \int_{-\infty}^{\infty} \sin(t-\sigma)u(t-2\sigma)x(\sigma)\,d\sigma$$

$\boxed{3.}$

$$h(t;\tau) = T[\delta(t-\tau)] = \int_{-\infty}^{\infty} \sin(t-\sigma)u(t-2\sigma)\delta(t-\tau)\,d\sigma$$

$$= \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)\delta(t-\tau)\,d\sigma$$ great!

$$= \sin(t-\tau)\,\underline{u(t-2\tau)}$$

Confirms time varying, not strictly difference $t-\tau$

c) $T[u(t)-u(t-1)] = T[u(t)u(1-t)]$

$$= \int_{-\infty}^{\infty} \sin(t-\tau)u(t-2\tau)u(\tau)u(1-\tau)\,d\tau$$

$\boxed{5}$

$$= \int_{0}^{1} \sin(t-\tau)\,d\tau \quad \text{if} \quad 1 < t/2 \Rightarrow 2 < t$$

$$= \cos(t-\tau)\Big|_0^1 = \cos(t-1) - \cos(t)$$

$$0 < t/2 < 1 \Rightarrow 0 < t < 2$$

$$= \int_0^{t/2} \sin(t-\tau)\,d\tau \quad \text{if}$$

$$= \cos(t-\tau)\Big|_0^{t/2} = \cos(t/2) - \cos(t)$$

$$= 0 \quad \text{if} \quad t < 0$$

3. Let $\mathcal{S}$ be a linear, time-invariant, and causal system. We know that the output corresponding to $x(t) = (t-3)\mathrm{u}(t-3)$ is

$$y(t) = 4\mathrm{e}^{-(t-3)}\mathrm{u}(t-3) - (t-4)\mathrm{u}(t-4).$$

(a) (3 points) What is the impulse response of $\mathcal{S}$? (Hint: consider that $\frac{\mathrm{d}}{\mathrm{d}t}t\mathrm{u}(t) = \mathrm{u}(t)$.)

(b) (5 points) Compute the output of $\mathcal{S}$ when $x(t) = (1-t)\mathrm{u}(t)$.

a) S is time invariant. Therefore, $T[t\,u(t)] = 4e^{-t}u(t) - (t-1)u(t-1)$

$$T[u(t)] = T\left[\frac{d}{dt}(t\,u(t))\right] = \frac{d}{dt}\left[4e^{-t}u(t) - (t-1)u(t-1)\right]$$

$$= -4e^{-t}u(t) + 4e^{-t}\delta(t) - \left[u(t-1) + (t-1)\delta(t-1)\right]$$

$$= -4e^{-t}u(t) + 4 - u(t-1)$$

$$T[\delta(t)] = T\left[\frac{d}{dt}u(t)\right] = \frac{d}{dt}\left[-4e^{-t}u(t) + 4 - u(t-1)\right]$$

$$= 4e^{-t}u(t) - 4e^{-t}\delta(t) - \delta(t-1)$$

$$= 4e^{-t}u(t) - 4 - \delta(t-1)$$

$$h(t-\tau;0) = T[\delta(t-\tau)] = 4e^{-(t-\tau)}u(t-\tau) - 4 - \delta(t-\tau-1)$$

b) $x(t) = u(t) - t\,u(t)$

$$T[x(t)] = T[u(t)] - T[t\,u(t)]$$

$$= -4e^{-t}u(t) + 4 - u(t-1) - \left(4e^{-t}u(t) - (t-1)u(t-1)\right)$$

$$= -8e^{-t}u(t) + 4 - u(t-1) + (t-1)u(t-1)$$

$$= -8e^{-t}u(t) + 4 + (t-2)u(t-1)$$

4. A system $\mathcal{S}_1$ is described by

$$\frac{dy}{dt} + 2y(t) = \frac{dx}{dt} - 2x(t), \quad t > 0, x(0) = 0, y(0) = 0.$$

(a) (3 points) Write down the impulse response function $h(t;\tau)$ of $\mathcal{S}_1$.

(b) (3 points) System $\mathcal{S}_1$ is now cascaded with a second linear, time-invariant system, $\mathcal{S}_2$ whose unit step response, $g_2(t)$ is given by

$$g_2(t) = t\,e^{-t}u(t).$$

Compute the impulse response, $h_{1,2}(t;\tau)$, of the cascaded combination.

a) $e^{2t}\frac{dy}{dt} + 2e^{2t}y(t) = e^{2t}\frac{dx}{dt} - 2e^{2t}x(t)$

$$\frac{d\left(e^{2t}y(t)\right)}{dt} = e^{2t}\frac{dx}{dt} - 2e^{2t}x(t)$$

$$e^{2t}y(t) = \int_0^t e^{2\tau}\frac{dx}{d\tau} - 2e^{2\tau}x(\tau)\,d\tau$$

$$y(t) = \int_0^t e^{2\tau-2t}\frac{dx}{d\tau} - 2e^{2\tau-2t}x(\tau)\,d\tau$$

$$h(t;\tau) = T[\delta(t-\tau)] = \int e^{2\sigma-2t}\delta'(t-\tau) - 2e^{2\sigma-2t}\delta(\sigma-\tau)\,d\sigma = \int_0^t e^{2\sigma-2t}\delta'(t-\tau)\,d\sigma - 2e^{2\tau-2t}\int_0^t \delta(\sigma-\tau)\,d\sigma$$

$$= \int_0^t e^{2\sigma-2t}\delta'(\sigma-\tau)\,d\sigma - 2e^{2\tau-2t}$$

Integrated by parts

$$= e^{2\tau-2t}\delta(t-\tau) - \int_0^t \delta(\sigma-\tau)\,2e^{2\sigma-2t}\,d\sigma - 2e^{2\tau-2t}$$

$$= \delta(t-\tau) - 2e^{2\tau-2t}\int_0^t \delta(\sigma-\tau)\,d\sigma - 2e^{2\tau-2t}$$

$$= \delta(t-\tau) - 4e^{-2(t-\tau)}$$

Also solved with Laplace to check:

$$s\mathcal{L}[y] + 2\mathcal{L}[y] = s\mathcal{L}[x] - 2\mathcal{L}[x]$$

$$\mathcal{L}[y] = \mathcal{L}[x]\frac{(s-2)}{(s+2)} = \mathcal{L}[x]\left(\frac{s+2}{s+2} - \frac{4}{s+2}\right)$$

$$h(t) = \mathcal{L}^{-1}\left[1 - \frac{4}{s+2}\right] = \delta(t) - 4e^{-2t}$$

b on next page

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b) $T_2[u(t)] = te^{-t}u(t)$

$T_2\{\delta(t)] = \frac{d(te^{-t}u(t))}{dt} = e^{-t}u(t) - te^{-t}u(t) + te^{-t}\delta(t)$

$= (e^{-t} - te^{-t})u(t)$ ✓

$h_2(t-T;0) = e^{-t+T} - (t-T)e^{-t+T}$

$h(t;T)$ from S_1: $\delta(t-T) - 4e^{-2(t-T)}$

$T_2[\delta(t-T) - 4e^{-2(t-T)}] =$

Must convolve impulse response with impulse response of 1st system

$= e^{-t+T} - (t-T)e^{-t+T} - \int_{-\infty}^{t} (e^{-\sigma} - \sigma e^{-\sigma})\, 4e^{-2(t-\sigma-T)}$

$= e^{-t+T} - (t-T)e^{-t+T} - \int_{-\infty}^{t} 4e^{\sigma-2t+2T} - 4\sigma e^{\sigma-2t+2T}\, d\sigma$

$= e^{-t+T} - (t-T)e^{-t+T} - 4e^{-2t+2T} \int_{-\infty}^{t} (1-\sigma)e^{\sigma} \Big|_{-\infty}^{t+5}$

−9.5