# UCLA

# DEPARTMENT OF ELECTRICAL ENGINEERING

**Winter 2008**

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(Fill in the following lines — if applicable)

NAME OF BEST FRIEND ON YOUR LEFT:

NAME OF BEST FRIEND ON YOUR RIGHT:

# EE102: SYSTEMS & SIGNALS

February 6, 2008

# MIDTERM EXAMINATION SOLUTIONS

**Instructions:**

**(i) Closed Book, Calculators Are NOT Allowed**

**(ii) Please bring your own LOOSE papers. Write On ONE Side Only (-5 points for each two-sided page)**

**(iii) Staple These Examination Papers With Your Papers — using your own stapler**

**(iv) Questions are equally weighted**

## QUESTION 1

### t-Domain Analysis

(Do NOT use Laplace Transforms)

**(i)** The IPOP relation of a SISO system $S$ is:

$$\mathbf{x}(\mathbf{t}) \longrightarrow [\mathbf{S}] \longrightarrow \mathbf{y}(\mathbf{t})$$

$$\mathbf{y}(\mathbf{t}) = \int_{-\infty}^{\mathbf{t}} \mathbf{x}(\tau)\,\mathbf{d}\tau + \int_{\mathbf{t}}^{\infty} \mathbf{e}^{\mathbf{t}}\,\mathbf{e}^{-\tau}\,\mathbf{x}(\tau)\,\mathbf{d}\tau, \quad \mathbf{t} \in (-\infty, \infty)$$

Find the IRF $\mathbf{h}(\mathbf{t}, \tau)$ of $S$, then state all properties of the system (no proof is required).

Find the USR (Unit Step Response) $\mathbf{g}(\mathbf{t})$ of $S$.

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**SOLS (i)**

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\,d\tau + \int_{t}^{\infty} e^{t}\,e^{-\tau}\,x(\tau)\,d\tau, \\ &= \int_{-\infty}^{\infty} [U(t-\tau) + e^{(t-\tau)}U\big(-(t-\tau)\big)]\,x(\tau)\,d\tau \\ \therefore \quad h(t-\tau) &= U(t-\tau) + e^{(t-\tau)}U\big(-(t-\tau)\big) \quad (\text{by BT}) \\ \therefore \quad \underline{h(t)} &= \underline{U(t) + e^{t}\,U(-t)} \end{aligned}$$

$S$: L, TI, NC

By definition:

$$\begin{aligned} g(t) &= \int_{-\infty}^{t} U(\tau)\,d\tau + \int_{t}^{\infty} e^{t}\,e^{-\tau}\,U(\tau)\,d\tau, \\ &= \int_{0}^{t} 1\,d\tau + e^{t}\int_{t}^{\infty} e^{-\tau}\,U(\tau)\,d\tau \\ \int_{0}^{t} 1\,d\tau &= t\,U(t) \\ e^{t}\int_{t}^{\infty} e^{-\tau}\,U(\tau)\,d\tau &= e^{t}\int_{0}^{\infty} e^{-\tau}\,1\,d\tau = e^{t}, \quad \text{for} \quad t < 0 \\ &= e^{t}\int_{t}^{\infty} e^{-\tau}\,1\,d\tau = 1, \quad \text{for} \quad t \geq 0 \end{aligned}$$

Finally

$$\mathbf{g(t) = (1+t)U(t) + e^{t}\,U(-t)}$$

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**(ii)** Compute the OP $y(t)$ of $S$ given that its input $x(t)$ is

$$\mathbf{x(t) = U(t-4)}, \quad \mathbf{t} \in (-\infty, \infty)$$

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**SOLS (ii)**

Since $S$ is TI we can have

$$\mathbf{x(t) = U(t-4)} \longrightarrow [\mathbf{S}] \longrightarrow \mathbf{g(t-4)} := \mathbf{y(t)}$$

Therefore

$$\mathbf{y(t)} = g(t-4) = \mathbf{(1+[t-4])U(t-4) + e^{(t-4)}\,U(-[t-4])}$$

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**(iii)** (<u>A Stand-Alone Problem</u>)

Systems $S_1$ and $S_2$ are described by the following diagrams:

$$\mathbf{S_1}: \quad \delta(\mathbf{t}-\sigma) \ \rightarrow [\mathbf{S_1 : L, TI}] \rightarrow \ (\mathbf{t}-\sigma)\,\mathbf{U}(\mathbf{t}-\sigma) \ \Rightarrow \ h_1(t) = t\,U(t)$$

and

$$\mathbf{S_2}: \quad \mathbf{U}(\mathbf{t}-\tau) \ \rightarrow \ [\mathbf{S_2 : L, TI}] \ \rightarrow \ \mathbf{e}^{-(\mathbf{t}-\tau)}\,\mathbf{U}(\mathbf{t}-\tau) \ \Rightarrow \ g_2(t) = e^{-t}\,U(t)$$

Your problem is to compute the OP $y(t)$ in the diagram below:

$$\mathbf{U(t)} \ \rightarrow [\mathbf{S_1}] \rightarrow [\mathbf{S_2}] \rightarrow \mathbf{y(t)} = ?$$

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**SOLS (iii)**

It follows from:

$$\mathbf{U(t)} \ \rightarrow [\mathbf{S_1}] \rightarrow [\mathbf{S_2}] \rightarrow \mathbf{y(t)} = ?$$

that, by commuting $S_1$ and $S_2$ — a legitimate move — you get

$$\mathbf{U(t)} \ \rightarrow [\mathbf{S_2}] \rightarrow [\mathbf{S_1}] \rightarrow \mathbf{y(t)} = ?$$

or:

$$U(t) \;\rightarrow [S_2] \rightarrow g_2(t) \rightarrow [S_1] \rightarrow y(t) = \int_{-\infty}^{\infty} h_1(t-\sigma)U(t-\sigma)\, g_2(\sigma)\, U(\sigma)\, d\sigma$$

It then follows from what you have done above that

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h_1(t-\sigma)U(t-\sigma)\, g_2(\sigma)\, U(\sigma)\, d\sigma \\
&= \int_{0}^{t} h_1(t-\sigma)\, g_2(\sigma)\, d\sigma, \quad t \geq 0 \\
&= \int_{0}^{t} (t-\sigma)\, e^{-\sigma}\, d\sigma, \quad t \geq 0 \\
\mathbf{y(t)} &= [\mathbf{-1 + t + e^{-t}}]\, \mathbf{U(t)}
\end{aligned}$$

## QUESTION 2

**s-Domain Analysis**

(<u>This is where Laplace Transforms shine</u>)

**(Stand-Alone Problems)**

**(i)** Let $S_1$ and $S_2$ be L,TI,C systems with IRF's $\mathbf{h_1(t)U(t)}$ and $\mathbf{h_2(t)U(t)}$, respectively.

Let $S_{12}$ denote the cascaded combination:

$$\mathbf{S_{12}}: \quad \longrightarrow [\mathbf{S_1}] \longrightarrow [\mathbf{S_2}] \longrightarrow$$

Show that the System Function $\mathbf{H_{12}(s)}$ of $S_{12}$ is

$$\mathbf{H_{12}(s) = H_1(s)\, H_2(s)}$$

where $H_1(s)$ and $H_2(s)$ are System Functions of $S_1$ and $S_2$, respectively.

Compute the USR $\mathbf{G_{12}(s)}$ of $\mathrm{S}_{12}$ — in terms of the System Functions $H_1(s)$ and $H_2(s)$.

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**SOLS (i)**

We have in the $s$-Domain:

$$1 \quad \longrightarrow [S_1] \longrightarrow \; H_1(s) \longrightarrow [S_2] \longrightarrow H_2(s)\, H_1(s) := H_{12}(s)$$

and

$$\frac{1}{s} \quad \longrightarrow [S_1] \longrightarrow \; G_1(s) \longrightarrow [S_2] \longrightarrow H_2(s)\, G_1(s) := G_{12}(s)$$

But

$$G_1(s) = H_1(s)\,\frac{1}{s} \;\Rightarrow\; \mathbf{G_{12}(s) = H_2(s) H_1(s) \frac{1}{s}} = H_{12}(s)\frac{1}{s} \; (\Leftarrow \; [\text{So So Cute}])$$

or

$$G_{12}(s) = H_2(s)\, G_1(s) = H_1(s)\, G_2(s) \qquad \text{(Double Cute)}$$

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**(ii)** Given:

$$\mathbf{x(t)} \;\longrightarrow\; [\mathbf{S}] \;\longrightarrow\; \mathbf{y(t)}$$

$$\mathbf{y(t)} = \int_{-\infty}^{\mathbf{t}} [\mathbf{U(t-\tau) + e^{-(t-\tau)}}]\ \mathbf{x(\tau)\ d\tau}, \quad \mathbf{t} \in (-\infty, \infty)$$

Find the OP $y(t)$ given that

$$\mathbf{x(t) = t\,U(t)}$$

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**SOLS (ii)**

First, you see that

$$h(t-\tau) = U(t-\tau) + e^{-(t-\tau)}U(t-\tau) \ \Rightarrow\ h(t) = U(t) + e^{-t}U(t)$$

$$\Rightarrow \quad H(s) = \frac{1}{s} + \frac{1}{s+1}$$

and

$$X(s) = \frac{1}{s^2}$$

Therefore

$$Y(s) = H(s)\,X(s) = \{\frac{1}{s} + \frac{1}{s+1}\}\,\frac{1}{s^2} = \frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\therefore \quad y(t) = [\frac{t^2}{2} + t - 1 + e^{-t}]\,U(t) = [\mathbf{-1 + t + \frac{t^2}{2} + e^{-t}}]\,\mathbf{U(t)}$$

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**(iii)** Consider the following systems:

$$\mathbf{S_1}: \qquad \delta(\mathbf{t}-\sigma) \ \rightarrow [\mathbf{S_1 : L, TI}] \rightarrow \ (\mathbf{t}-\sigma)\,\mathbf{U(t}-\sigma)$$

and

$$\mathbf{S_2}: \qquad \mathbf{U(t-\tau)} \ \rightarrow \ [\mathbf{S_2 : L, TI}] \ \rightarrow \ \mathbf{e^{-(t-\tau)}\,U(t-\tau)}$$

Your problem is to compute the OP $y(t)$ of the following cascaded system

$$\mathbf{U(t)} \ \rightarrow [\mathbf{S_1}] \rightarrow [\mathbf{S_2}] \rightarrow \ \mathbf{y(t)} = ?$$

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**SOLS (iii)**

You have from the above:

$$\mathbf{S_1}: \qquad \delta(\mathbf{t}) \;\rightarrow [\mathbf{S_1}:\mathbf{L},\mathbf{TI}] \rightarrow \; \mathbf{t}\,\mathbf{U}(\mathbf{t}) := h_1(t) \;\Rightarrow\; H_1(s) = \frac{1}{s^2}$$

and

$$\mathbf{S_2}: \qquad \mathbf{U}(\mathbf{t}) \;\rightarrow\; [\mathbf{S_2}:\mathbf{L},\mathbf{TI}] \;\rightarrow\; \mathbf{e^{-t}}\,\mathbf{U}(\mathbf{t}) := g_2(t) \;\Rightarrow\; G_2(s) = \frac{1}{s+1}$$

and

$$\mathbf{U}(\mathbf{t}) \;\rightarrow [\mathbf{S_1}] \rightarrow [\mathbf{S_2}] \rightarrow \; \mathbf{y}(\mathbf{t}) = \mathbf{g_{12}}(\mathbf{t})$$

Therefore, by your results in Q 2-(i):

$$G_{12}(s) = H_2(s)H_1(s)\frac{1}{s} = H_1(s)G_2(s) \; (\Leftarrow \; [\textit{So So Cute}])$$

we get

$$G_{12}(s) = \frac{1}{s^2}\,\frac{1}{s+1} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \;\Rightarrow\; \mathbf{y}(\mathbf{t}) = [-\mathbf{1} + \mathbf{t} + \mathbf{e^{-t}}]\,\mathbf{U}(\mathbf{t})$$

Alternately: For:

$$S_1: \quad h_1(t) = t\,U(t) \quad \Rightarrow \quad H_1(s) = \frac{1}{s^2}$$

For:

$$S_2: \quad g_2(t) = e^{-t}\,U(t) \quad \Rightarrow \quad G_2(s) = \frac{1}{s+1}$$

Now we know that, by switching $S_1$ and $S_2$:

$$\mathbf{U}(\mathbf{t}) \;\rightarrow [\mathbf{S_1}] \rightarrow [\mathbf{S_2}] \rightarrow \mathbf{y}(\mathbf{t}) \quad \Leftrightarrow \quad \mathbf{U}(\mathbf{t}) \;\rightarrow [\mathbf{S_2}] \rightarrow [\mathbf{S_1}] \rightarrow \mathbf{y}(\mathbf{t})$$

Therefore

$$\Leftrightarrow \quad \mathbf{U}(\mathbf{t}) \;\rightarrow [\mathbf{S_2}] \rightarrow \; \mathbf{g_2}(\mathbf{t}) \rightarrow \; [\mathbf{S_1}] \rightarrow \mathbf{y}(\mathbf{t})$$

$$\therefore \quad Y(s) = H_1(s)\,G_2(s) = \frac{1}{s^2}\,\frac{1}{s+1}$$

$\therefore$ as found before:

$$\mathbf{y}(\mathbf{t}) = [-\mathbf{1} + \mathbf{t} + \mathbf{e^{-t}}]\,\mathbf{U}(\mathbf{t}) \quad (\textit{Ha Ha, So Cute!})$$

**QUESTION 3**

**t-Domain and or s-Domain**

**(Stand-Alone Problems)**

**(i)** A system $S$ :

$$\mathbf{x(t)} \longrightarrow [\mathbf{S}] \longrightarrow \mathbf{y(t)}$$

is described by the differential equation:

$$\begin{aligned} 2\frac{dy(t)}{dt} + 4y(t) &= \frac{dx(t)}{dt}, \quad t > 0, \\ y(0) &= 0 \\ x(0) &= 0 \end{aligned}$$

(i) Find the IRF $\mathbf{h(t)}$ of $S$.
(ii) Find $\mathbf{y(t)}$ given that $\mathbf{x(t)} = \sin \mathbf{t}\, \mathbf{U(t)}$.

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**SOLS (i)**

We have

$$2\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} \quad \Rightarrow \quad \frac{dy(t)}{dt} + 2y(t) = \frac{1}{2}\frac{dx(t)}{dt}$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} = \frac{s}{2(s+2)} = \frac{1}{2} - \frac{1}{s+2} \quad \Rightarrow \quad \mathbf{h(t)} = \frac{\mathbf{1}}{\mathbf{2}}\,\delta(\mathbf{t}) - \mathbf{e^{-2t}}\,\mathbf{U(t)}$$

(ii)

$$Y(s) = H(s)X(s) = \frac{s}{2(s+2)}\,\frac{1}{s^2+1} = \frac{-1}{5(s+2)} + \frac{s}{5(s^2+1)} + \frac{1}{10(s^2+1)}$$

$$\therefore \quad \mathbf{y(t)} = [-\frac{\mathbf{1}}{\mathbf{5}}\mathbf{e^{-2t}} + \frac{\mathbf{1}}{\mathbf{5}}\cos\mathbf{t} + \frac{\mathbf{1}}{\mathbf{10}}\sin\mathbf{t}]\,\mathbf{U(t)}$$

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**(ii)** Find

$$\mathcal{L}_{\mathbf{s}}\Big\{\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{-t}}\, \cos(\mathbf{t}-\tau)\, \cos\tau\, \mathbf{d}\tau, \quad \mathbf{t} \geq \mathbf{0}\Big\}$$

and

$$\mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\tau} \cos(\mathbf{t}-\tau) \cos\tau \, \mathbf{d}\tau, \quad \mathbf{t} \geq \mathbf{0}\Big\}$$

**SOLS (ii)**

We have

$$\mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\mathbf{t}} \cos(\mathbf{t}-\tau) \cos\tau \, \mathbf{d}\tau, \quad \mathbf{t} \geq \mathbf{0}\Big\} = \mathcal{L}_{\mathbf{s}}\Big\{\mathbf{e}^{-\mathbf{t}}\mathbf{f}(\mathbf{t})\Big\}$$

where

$$f(t) := \int_0^t \cos(t-\tau) \cos\tau \, d\tau$$

Now

$$F(s) = \mathcal{L}_s\{\cos t\}\, \mathcal{L}_s\{\cos t\} = [\frac{s}{s^2+1}]^2$$

∴

$$\mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\mathbf{t}} \cos(\mathbf{t}-\tau) \cos\tau \, \mathbf{d}\tau, \quad \mathbf{t} \geq \mathbf{0}\Big\} = \mathbf{F}(\mathbf{s}+\mathbf{1}) = [\frac{\mathbf{s}+\mathbf{1}}{(\mathbf{s}+\mathbf{1})^{\mathbf{2}}+\mathbf{1}}]^{\mathbf{2}}$$

Next

$$\mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\tau} \cos(\mathbf{t}-\tau) \cos\tau \, \mathbf{d}\tau\Big\} = \mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \cos(\mathbf{t}-\tau)\, [\mathbf{e}^{-\tau} \cos\tau]\, \mathbf{d}\tau\Big\}$$

∴

$$\begin{aligned} \mathcal{L}_{\mathbf{s}}\Big\{ \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\tau} \cos(\mathbf{t}-\tau) \cos\tau \, \mathbf{d}\tau\Big\} &= \mathcal{L}_s\{\cos t\}\, \mathcal{L}_s\{e^{-t} \cos t\} \\ &= \frac{\mathbf{s}}{\mathbf{s}^{\mathbf{2}}+\mathbf{1}} \, \frac{\mathbf{s}+\mathbf{1}}{(\mathbf{s}+\mathbf{1})^{\mathbf{2}}+\mathbf{1}} \end{aligned}$$

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**(iii)** Let $S$ be a L,TI,C system described by the IPOP relation:

$$\mathbf{x}(\mathbf{t}) \longrightarrow [\mathbf{S}] \longrightarrow \mathbf{y}(\mathbf{t})$$

$$\mathbf{y}(\mathbf{t}) = \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{-\sigma}\, \mathbf{x}(\mathbf{t}-\sigma)\, \mathbf{d}\sigma, \quad \mathbf{t} \geq \mathbf{0}.$$

Write down the IRF $\mathbf{h}(\mathbf{t})$ of $S$. Then compute its system function $\mathbf{H}(\mathbf{s})$ and its OP $\mathbf{y}(\mathbf{t})$ when the **IP**

$$\mathbf{x}(\mathbf{t}) = \sinh \mathbf{t}\, \mathbf{U}(\mathbf{t})$$

is applied to $S$.

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**SOLS (iii)**

We have

$$\begin{aligned}
y(t) &= \int_0^t e^{-\sigma}\, x(t-\sigma)\, d\sigma, \quad t \geq 0 \\
&= \int_0^t e^{-(t-\tau)}\, x(\tau)\, d\tau \\
\Rightarrow \quad h(t) &= e^{-t}\, U(t) \\
H(s) &= \frac{1}{s+1} \\
\therefore \quad Y(s) &= H(s)\, [X(s)] \\
&= \frac{1}{s+1}\, [\frac{1}{s^2-1}] \\
&= \frac{1}{2(s-1)} + \frac{-1}{2(s+1)^2} \\
\Rightarrow \quad \mathbf{y}(\mathbf{t}) &= [\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e}^{\mathbf{t}} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{t}\, \mathbf{e}^{-\mathbf{t}}]\, \mathbf{U}(\mathbf{t})
\end{aligned}$$

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