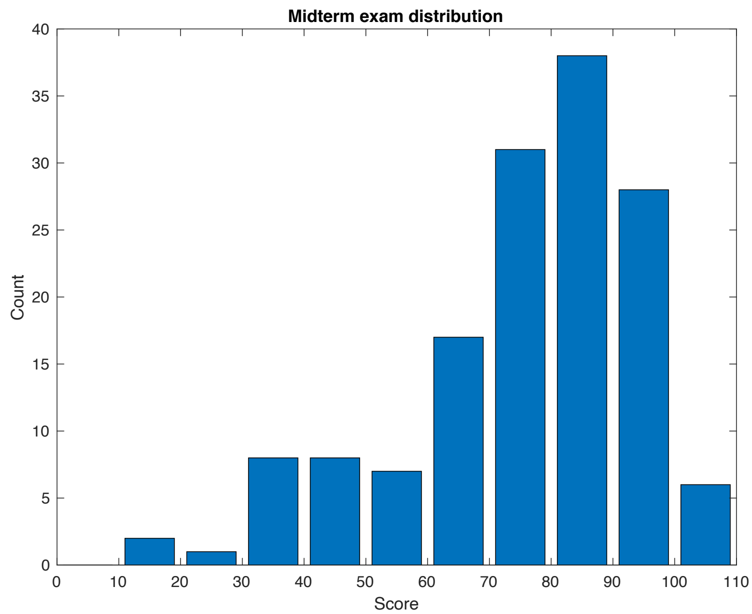
ECE 102, Fall 2021 | **MIDTERM STATISTICS**
Department of Electrical and Computer Engineering | Prof. J.C. Kao
University of California, Los Angeles | TAs T. Monsoor & S. Pei



Statistics: Mean: 74.2 (out of 100), Standard deviation: 19.3, Median: 78

Comments:

- If you have grading questions, please submit through Gradescope. Regrades should *only be submitted* if you believe we applied the rubric mistakenly. When we regrade, we re-evaluate the entire question, and so while rare, losing points is possible if we mis-graded your work.
- If you have particular questions about a midterm question and its grading, please see:
  - Q1a, Q2: Siyou
  - Q1b, Q3: Tonmoy
  - Bonus: Jonathan
- The variance of the midterm was higher than we expected. We were pleased that many students did solidly on the exam, however, we recognize that many students did not do as well.

  We are therefore making you the following offer: if you score better on the final than the midterm, we will replace your midterm score with your final score.

ECE102, Fall 2021 **Midterm Solution**

Department of Electrical and Computer Engineering Prof. J.C. Kao

University of California, Los Angeles TAs: T. Monsoor, S. Pei

UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 3 Nov 2021.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	____	/ 30
Problem 2	____	/ 35
Problem 3	____	/ 35
BONUS	____	/ 6 bonus points
Total	____	/ 100 points + 6 bonus points

1. **Signal and System Properties** (30 points).

(a) (15 points) *Signal Properties.*

i. (8 points) What are the real part and imaginary part of $f(t) = e^{j3\pi} + je^{1+j\frac{\pi}{4}}$?

Solution:

$$
\begin{aligned}
f(t) &= e^{j3\pi} + je^{1+j\frac{\pi}{4}} \\
&= -1 + je \cdot e^{j\frac{\pi}{4}} \\
&= -1 + je\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) \\
&= -1 + je \cdot \frac{1}{\sqrt{2}} - e \cdot \frac{1}{\sqrt{2}} \\
&= -1 - \frac{e}{\sqrt{2}} + j\frac{e}{\sqrt{2}}
\end{aligned}
$$

The real part of $f(t)$ is $-1 - \frac{e}{\sqrt{2}}$, The imaginary part of $f(t)$ is $\frac{e}{\sqrt{2}}$.

ii. (7 points) For $x(t)$ indicated in the figure below, sketch $y(t) = x(t+2) + x(-2t+2)$.

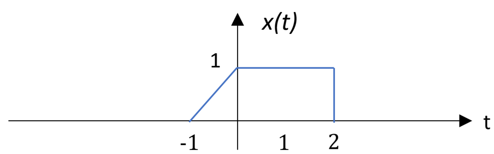


Figure 1: $x(t)$

Solution:
We can obtain $x(t+2)$ from $x(t)$ by first shifting $x(t)$ by 2 units to the left. We can obtain $x(-2t+2)$ from $x(t)$ by first shifting $x(t)$ by 2 units to the left and then flipping $x(t+2)$ and compressing $x(-t+2)$ by a factor of 2. By adding them together, we can obtain $y(t)$ in the figure below.

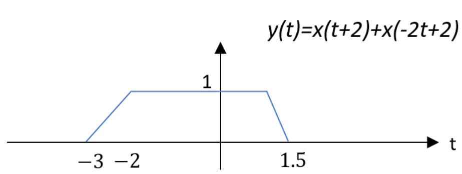


Figure 2: $y(t)$

(b) (15 points) *System Properties.*

i. (5 points) A system's impulse response is $h(t) = \frac{u(t-1)}{t^2}$. Is the system BIBO stable?

Solution: A system is BIBO stable if and only if:

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty$$

For the impulse response given in the problem,

$$\begin{aligned}\int_{-\infty}^{+\infty} |h(t)|dt &= \int_{-\infty}^{+\infty} \left|\frac{u(t-1)}{t^2}\right| dt \\ &= \int_{1}^{+\infty} \frac{1}{t^2} dt \\ &= 1\end{aligned}$$

Therefore, the system is BIBO stable.

ii. (5 points) The output of a system is given by $y(t) = \cos(2t) + 2$, when the input is $x(t) = \cos(t)$. Is the system LTI?

Solution: For the given system, the input contains frequency content at $\omega = 1$ rad/seconds but the output contains frequency content at $\omega = 0$ rad/seconds and $\omega = 2$ rad/seconds. Since complex exponentials are eigenfunctions of LTI systems, LTI systems cannot generate new frequency content (it can only scale and shift signals at the input frequency), therefore the given system is not LTI.

iii. (5 points) The input-output relationship of a system is given by:

$$\frac{dy(t)}{dt} = 2\sin[x(t-1)] + 3\cos[x(t-1)]$$

Is the system time-invariant?

Solution: Let,

$$z(t) = x(t - t_0)$$

Then,

$$\frac{d}{dt}\mathcal{S}[z(t)] = 2\sin[z(t-1)] + 3\cos[z(t-1)]$$

Plugging in the value of $z(t)$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{S}[x(t-t_0)] &= 2\sin[x(t-t_0-1)] + 3\cos[x(t-t_0-1)] \\ &= \frac{d}{dt}[y(t-t_0)]\end{aligned}$$

Therefore, the system is time-invariant.

2. **Eigenfunctions of LTI systems** (35 points).

As we know, $x(t) = e^{st}$ (where s is complex) are eigenfunctions of all LTI systems, i.e., $y(t) = e^{st} * h(t) = ae^{st}$ (where a is a complex constant) holds true for all $h(t)$. However, for some functions $\tilde{x}(t)$, we have $y(t) = a\tilde{x}(t)$ only for some $h_{specific}(t)$. To distinguish $\tilde{x}(t)$ from eigenfunctions, we call such an input a "semi-eigenfunction" in this question.

(a) (10 points) Determine if $x(t) = \sin(\omega_0 t)$ is an eigenfunction of an LTI system.

(**Hint:** As in lecture and on the HW, you should simplify the convolution integral $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$, where Euler's formula will help.)

Solution: Assume that $h(t)$ is the impulse response of the system. Then the output $y(t)$ to input $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$is as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \frac{1}{2j}\int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)}h(\tau)d\tau - \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)}h(\tau)d\tau \\ &= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0\tau}h(\tau)d\tau}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2} \end{aligned}$$

For $x(t)$ to be an eigenfunction for the system, its corresponding output should be of the form $ax(t)$, where a is constant. The output to $\sin(\omega_0 t)$ is:

$$y(t) = \frac{1}{2j}a_1 e^{j\omega_0 t} - \frac{1}{2j}a_2 e^{-j\omega_0 t}$$

Since, in general $a_1 \neq a_2$, we cannot construct again $\sin(\omega_0 t)$ in $y(t)$. Therefore $\sin(\omega_0 t)$ is not an eigenfunction for an LTI system.

(b) (10 points) Determine if $x(t) = \sin(\omega_0 t)$ is a "semi-eigenfunction" for an LTI system H_1 with $h_1(t) = \text{rect}(t)$. To be clear, in this question, $h_{specific}(t) = h_1(t)$.

Solution: The output $y(t)$ to input $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$is as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h_1(\tau)d\tau \\ &= \frac{1}{2j}\int_{-0.5}^{0.5} e^{j\omega_0(t-\tau)}d\tau - \frac{1}{2j}\int_{-0.5}^{0.5} e^{-j\omega_0(t-\tau)}d\tau \\ &= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{-0.5}^{0.5} e^{-j\omega_0\tau}d\tau}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-0.5}^{0.5} e^{j\omega_0\tau}d\tau}_{=a_2} \end{aligned}$$

The area beneath $e^{-j\omega_0\tau}$ is the same as the area beneath $e^{j\omega_0\tau}$, thus we have $a_1 = a_2$ Therefore, the output to $\sin(\omega_0 t)$ is:

$$y(t) = \frac{1}{2j}a_1 e^{j\omega_0 t} - \frac{1}{2j}a_1 e^{-j\omega_0 t} = a_1\sin(\omega_0 t)$$

and $\sin(\omega_0 t)$ is a "semi-eigenfunction" for the LTI system H_1.

(c) (15 points) $h_1(t) = \mathrm{rect}(t)$ is real and even. Show that $x(t) = \sin(\omega_0 t)$ is an "semi-eigenfunction" for all systems with a real and even impulse response, i.e., $h(t) = h(-t)$.

(**Hint**: Let $p = -\tau$, $\int_{-\infty}^{\infty} e^{-j\omega_0\tau}h(\tau)d\tau = \int_{-\infty}^{\infty} e^{-j\omega_0(-p)}h(-p)d(-p)$ where τ and p are dummy variables.)

Solution: The output $y_1(t)$ to input $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$is as follows:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \frac{1}{2j}\int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)}h(\tau)d\tau - \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)}h(\tau)d\tau \\
&= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0\tau}h(\tau)d\tau}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2}
\end{aligned}$$

Let $p = -\tau$ in the first integral, then

$$\begin{aligned}
&= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0(-p)}h(-p)d(-p)}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2} \\
&= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{\infty}^{-\infty} e^{j\omega_0 p}(-h(p))dp}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2}
\end{aligned}$$

Because $-\int_a^b x(t)dt = \int_b^a x(t)dt$,

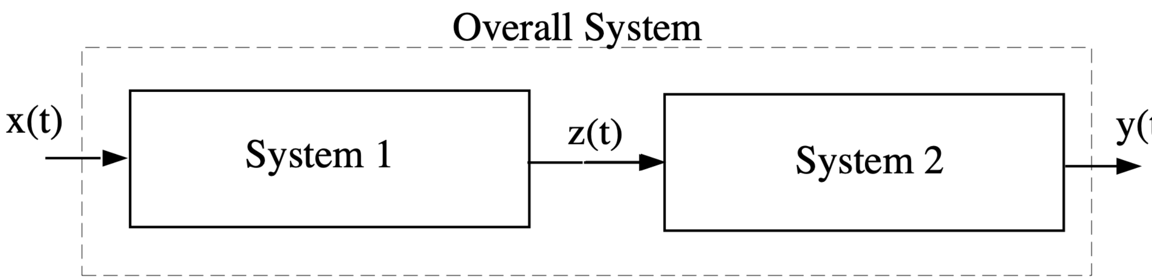
$$= \frac{1}{2j}e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0 p}h(p)dp}_{=a_1} - \frac{1}{2j}e^{-j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} e^{j\omega_0\tau}h(\tau)d\tau}_{=a_2}$$

Since p and τ are just dummy variables inside the integral, we have $a_1 = a_2$. Therefore, the output to $\sin(\omega_0 t)$ is:

$$y(t) = \frac{1}{2j}a_1 e^{j\omega_0 t} - \frac{1}{2j}a_1 e^{-j\omega_0 t} = a_1\sin(\omega_0 t)$$

and $\sin(\omega_0 t)$ is a "semi-eigenfunction" for the LTI systems with a real and even impulse response $h(t) = h(-t)$.

3. **LTI Systems and Fourier Series** (35 points).



Consider the series cascade of two LTI systems shown above. The two LTI systems are characterized by the following input-output relationships:

$$\textit{System 1}: z(t) = \int_{t-2}^{t} x(\tau)d\tau$$
$$\textit{System 2}: y(t) = \int_{t-4}^{t} z(\tau)d\tau$$

(a) (10 points) Compute the impulse response of the overall system.

Hint: You might find the following convolution result useful:

$$\text{rect}\left(\frac{t}{a} - 0.5\right) * \text{rect}\left(\frac{t}{b} - 0.5\right) = \begin{cases} 0, & t \leq 0 \\ t, & 0 \leq t \leq a \\ a, & a \leq t \leq b \\ a+b-t, & b \leq t \leq a+b \\ 0, & t \geq a+b \end{cases}$$

where $1 \leq a \leq b$.

Solution: Let the impulse response of *System 1* be $h_1(t)$ and the impulse response of *System 2* be $h_2(t)$. Then, by convolution theorem:

$$z(t) = x(t) * h_1(t)$$
$$y(t) = z(t) * h_2(t)$$

Hence,

$$\begin{aligned} y(t) &= [x(t) * h_1(t)] * h_2(t) \\ &= x(t) * [h_1(t) * h_2(t)] \end{aligned}$$

Therefore,

$$h_{overall}(t) = h_1(t) * h_2(t)$$

We can compute the impulse responses:

$$\begin{aligned} h_1(t) &= \int_{t-2}^{t} \delta(\tau) d\tau \\ &= u(t) - u(t-2) \\ &= \text{rect}\left(\frac{t}{2} - 0.5\right) \end{aligned}$$

Similarly,

$$\begin{aligned} h_2(t) &= \int_{t-4}^{t} \delta(\tau) d\tau \\ &= u(t) - u(t-4) \\ &= \text{rect}\left(\frac{t}{4} - 0.5\right) \end{aligned}$$

Then, using the result given in the problem ($a = 2, b = 4$):

$$h_{overall}(t) = \begin{cases} 0, & t \leq 0 \\ t, & 0 \leq t \leq 2 \\ 2, & 2 \leq t \leq 4 \\ 6-t, & 4 \leq t \leq 6 \\ 0, & t \geq 6 \end{cases}$$

(b) (10 points) Consider that the input to the overall system is the periodic signal $x(t)$ defined as:

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - 6k)$$

This signal is shown in Figure 3. Determine the Fourier series coefficients of $x(t)$, which we denote x_k, for $-\infty < k < +\infty$.

Solution: The period of the signal is $T_0 = 6$. We will find the Fourier series by integrating over the impulse at $t = 0$. The Fourier series x_k are:

$$\begin{aligned} x_k &= \frac{1}{6} \int_{-3}^{3} \delta(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{6} \int_{-3}^{3} \delta(t) e^{0} dt \\ &= \frac{1}{6} \end{aligned}$$

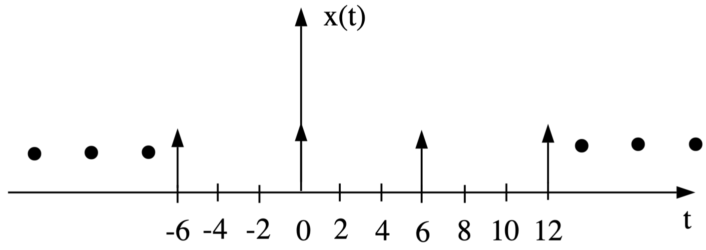


Figure 3: $x(t)$ for question 3b.

(c) (5 points) Determine the output, $y(t)$, of the overall system above to the periodic signal $x(t)$ in question 3b. You may leave your answer in terms of a function,

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-jk\omega_0 t}dt$$

where $h(t)$ was your answer to question 3a. You don't have to simplify this integral, or plug in $h(t)$ from question 3a, so that your answer here does not depend on your answer to 3a. If you are uncertain of your answer to 3b, you may leave the expression in terms of x_k.

Solution: Recall that $e^{j\omega t}$ are eigenfunctions of LTI systems:

$$\mathcal{S}[e^{j\omega t}] = H(j\omega)e^{j\omega t}$$

where,

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt.$$

Using the above result and the extended linearity property of LTI systems, we have

$$y(t) = \frac{1}{6}\sum_{k=-\infty}^{k=+\infty} H\left(jk\frac{\pi}{3}\right) e^{jk\frac{\pi}{3}t}$$

You could have also solved this by writing out the convolution integral.

$$\begin{aligned}
y(t) &= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\cdot\frac{1}{6}\sum_{k=-\infty}^{\infty} e^{jk\omega_0(t-\tau)}d\tau \\
&= \frac{1}{6}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-jk\omega_0\tau}d\tau \\
&= \frac{1}{6}\sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{3}t}\int_{-\infty}^{\infty} h(\tau)e^{-jk\frac{\pi}{3}\tau}d\tau \\
&= \frac{1}{6}\sum_{k=-\infty}^{\infty} H\left(jk\frac{\pi}{3}\right) e^{jk\frac{\pi}{3}t}
\end{aligned}$$

(d) (10 points) Consider the signal $w(t) = y(3t)$, where $y(t)$ is the periodic output defined in question 3c. What is the period of $w(t)$? Determine the Fourier series coefficients for $w(t)$, denoted w_k, for $-\infty < k < +\infty$.

Solution: If we denote the Fourier series coefficients of $y(t)$ by y_k, then using the expression from previous part we have:

$$y_k = \frac{1}{6} H\left(jk\frac{\pi}{3}\right), \text{for all } \ k$$

As in HW #4, we know that the time-scaled signal $w(t) = y(3t)$ will have the same Fourier coefficients as $y(t)$, but be spaced at a different frequency (since the period has changed). While $y(t)$ had a period of $T_0 = 6$ and thus $\omega_0 = \pi/3$, $w(t)$ has a period of $T_w = 2$ and thus $\omega_w = \pi$.

$$\begin{aligned} w_k &= y_k \\ &= \frac{1}{6} H\left(jk\frac{\pi}{3}\right), \text{for all } \ k \end{aligned}$$

Bonus Question (6 points) Two functions $f(t)$ and $g(t)$ have Fourier series coefficients f_k and g_k, respectively, and

$$g_k = (-1)^k f_k$$

Find a simple expression for $g(t)$ in terms of $f(t)$. Assume $g(t)$ and $f(t)$ have the same period.

Solution: We will write the Fourier series for $g(t)$:

$$\begin{aligned} g(t) &= \sum_{k=-\infty}^{\infty} g_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} (-1)^k f_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} e^{-j\pi k} f_k e^{jk\omega_0 t} \end{aligned}$$

We next note that we want to combine the exponential terms. We must therefore find a τ such that

$$\omega_0 \tau = \pi$$

which occurs at $\tau = T_0/2$. Hence,

$$\begin{aligned} g(t) &= \sum_{k=-\infty}^{\infty} e^{-j\pi k} f_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} e^{-jk\omega_0 \cdot \frac{T_0}{2}} f_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 \left(t - \frac{T_0}{2}\right)} \\ &= f(t - T_0/2) \end{aligned}$$

We therefore have that

$$\boxed{g(t) = f(t - T_0/2)}$$