ECE102, Fall 2020 **Midterm Solution**
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UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Monday, 9 Nov 2020.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Do not send a copy of this exam to any classmates or discuss the exam on Piazza until after Wednesday, 11 Nov 2020. There are students who will take the exam at different times.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 30
Problem 2	_____	/ 40
Problem 3	_____	/ 30
BONUS	_____	/ 6 bonus points
Total	_____	/ 100 points + 6 bonus points

1. **Signal and System Properties** (30 points).

   (a) (6 points) *Complex number.* What are the phase and amplitude of the following number?

$$x = (1 + j)e^{3j} \tag{1}$$

   Solution:

$$x = \sqrt{2}e^{j\frac{\pi}{4}}e^{3j} = \sqrt{2}e^{(\frac{\pi}{4}+3)j}$$

   Hence, the amplitude is $\sqrt{2}$, the phase is $3 + \frac{\pi}{4}$

   (b) (6 points) If $x(t)$ is an even function, and $x(t-1)$ is also even, is $x(t)$ periodic? Explain your reasoning.

   Solution:
   As $x(t)$ and $x(t-1)$ are both even, we have $x(t) = x(-t)$ and $x(t-1) = x(-t-1)$. If we substitute the $t-1$ with t', we have $x(-t') = x(t') = x(-t-1) = x(-(t-1)-2) = x(-t'-2)$. Thus we can see that $x(t)$ is periodic.

   (c) (6 points) Evaluate the expression

$$\int_0^\infty \delta(t-1)(t-2)^2 dt.$$

   Solution:
   Recall that according to the **sifting property**,

$$f(t)\delta(t-1) = f(1)\delta(t-1).$$

   In this case, $f(t) = (t-2)^2$; therefore,

$$(1-2)^2\delta(t-1) = (1)^2\delta(t-1) = \delta(t-1).$$

   Finally, we integrate the above expression,

$$\int_0^\infty (t-2)^2\delta(t-1)dt = 1\int_0^\infty \delta(t-1) = 1.$$

   (d) (12 points) Consider the system with input $x(t)$ and output $y(t)$:

$$y(t) = \int_{-\infty}^{t} x(\lambda)u(t+1)d\lambda.$$

   Determine if the system is:

i. Linear
ii. Time-invariant
iii. Causal
iv. Stable

You must justify your answer to receive full credit.

Solution:
Linearity: **Linear** We will check here homogeneity and superposition:

$$\begin{aligned} u(t+1)\int_{-\infty}^{t}(ax_1(\lambda)+bx_2(\lambda))\,d\lambda &= u(t+1)\int_{-\infty}^{t}ax_1(\lambda)d\lambda + u(t+1)\int_{-\infty}^{t}bx_2(\lambda)d\lambda \\ &= a\left(u(t+1)\int_{-\infty}^{t}x_1(\lambda)d\lambda\right)+b\left(u(t+1)\int_{-\infty}^{t}x_2(\lambda)d\lambda\right) \\ &= ay_1(t)+by_2(t) \end{aligned}$$

The system is then linear.
Time-invariance: **Time-varint** If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = u(t+1)\int_{-\infty}^{t}x_\tau(\lambda)d\lambda = u(t+1)\int_{-\infty}^{t}x(\lambda-\tau)d\lambda$$

Let $\lambda' = \lambda - \tau$, then

$$y_\tau(t) = u(t+1)\int_{-\infty}^{t-\tau}x(\lambda')d\lambda'$$

On the other hand,

$$y(t-\tau) = u(t+1-\tau)\int_{-\infty}^{t-\tau}x(\lambda)d\lambda$$

Therefore $y(t-\tau) \neq y_\tau(t)$. The system is then time variant.
Causality: **Causal** The system is integrating values of $x(t)$ up to time t. The output does not depend on future values of $x(t)$, the system is then causal.
Stability: **Unstable** Even if $x(t)$ is absolutely bounded, the integral:

$$\int_{-\infty}^{t}x(\lambda)d\lambda$$

cannot in general be bounded, the system is unstable. For instance, suppose $x(t) = 1$, then $\int_{-\infty}^{t}1d\lambda \to \infty$. Another example, suppose $x(t) = u(t)$, then

$$y(t) = u(t+1)\int_{-\infty}^{t}u(\lambda)d\lambda = u(t+1)\int_{0}^{t}1d\lambda = u(t+1)t$$

$u(t+1)t$ cannot be bounded as $t \to \infty$, because $u(t+1)t \to \infty$ as $t \to \infty$.

2. **Impulse response, LTI systems and Convolution** (40 points).

Note: Parts (a) and (b) of this problem are independent. In part (a) of the problem, you might find the following convolution result useful

$$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

(a) Consider the LTI system characterized by the Input/Output relationship:

$$\text{System 1 :} \qquad y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}x(\tau)d\tau$$

i. (5 points) Write the impulse response of the system, $h_1(t)$.

Solution: Impulse response is the output of the system when the input is the dirac delta, $\delta(t)$

$$h_1(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}\delta(\tau)d\tau$$

Using the sifting property of the impulse,

$$h_1(t) = e^{-2t}\int_{-\infty}^{t} \delta(\tau)d\tau$$
$$h_1(t) = e^{-2t}u(t)$$

ii. (10 points) Determine a closed-form expression for the output (y(t)) of System 1 for the input

$$x(t) = 3e^{-4t}u(t-2)$$

Solution: Let's first compute the output of the system when the input is $e^{-4t}u(t)$

$$y_1(t) = e^{-4t}u(t) * e^{-2t}u(t)$$

Using the convolution result,

$$y_1(t) = -\frac{1}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t)$$

We can rewrite the input $x(t)$ as follows

$$x(t) = 3e^{-4(t-2)}u(t-2) \times e^{-8}$$

Then using the linearity and time-invariance property of system 1,

$$\begin{aligned} y(t) &= 3e^{-8}y_1(t-2) \\ &= 3e^{-8}\{-\frac{1}{2}e^{-4(t-2)}u(t-2) + \frac{1}{2}e^{-2(t-2)}u(t-2)\} \end{aligned}$$

iii. (10 points) Determine a closed-form expression for the output (y(t)) of System 1 for the input

$$x(t) = \{u(t) - u(t-3)\}$$

Solution: The output of an LTI system is given by the convolution

$$\begin{aligned} y(t) &= \{u(t) - u(t-3)\} * e^{-2t}u(t) \\ &= e^{-2t}u(t) * u(t) - e^{-2t}u(t) * u(t-3) \end{aligned}$$

Using the convolution result,

$$\begin{aligned} y_2(t) &= e^{-2t}u(t) * u(t) \\ &= -\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) \end{aligned}$$

Since the system is time-invariant,

$$\begin{aligned} y(t) &= y_2(t) - y_2(t-3) \\ &= \{-\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t)\} - \{-\frac{1}{2}e^{-2(t-3)}u(t-3) + \frac{1}{2}u(t-3)\} \end{aligned}$$

iv. (5 points) Determine the output of System 1, y(t), for the input

$$x(t) = \delta(t) - \delta(t-3)$$

Solution: We know that the output of an LTI system is given by the convolution of the input and the impulse response

$$\begin{aligned} y(t) &= x(t) * h_1(t) \\ &= \{\delta(t) - \delta(t-3)\} * h_1(t) \\ &= \{\delta(t) * h_1(t\}) - \{\delta(t-3) * h_1(t)\} \\ &= h_1(t) - h_1(t-3) \\ y(t) &= e^{-2t}u(t) - e^{-2(t-3)}u(t-3) \end{aligned}$$

(b) If $y(t) = x(t) * h(t)$ is the output of an LTI system with input $x(t)$ and impulse response $h(t)$, then show the following properties. Note, you will not receive credit if you simply state that the differentiator and convolution are LTI systems and can be interchanged. Although this is true, we want you to work with the convolution integral.

Hint: You may interchange the order of integration and differentiation.

i. (5 points) $\frac{d}{dt}y(t) = x(t) * (\frac{d}{dt}h(t))$

Solution: We know that the output of the LTI system is given by

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

Now, taking the derivative with respect to t of both sides

$$\frac{d}{dt}y(t) = \frac{d}{dt}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Assuming that the functions are sufficiently smooth, the derivative can be pulled through the integral

$$\begin{aligned} \frac{d}{dt}y(t) &= \int_{-\infty}^{\infty} \frac{d}{dt}\{x(\tau)h(t-\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\frac{d}{dt}h(t-\tau)d\tau \\ \frac{d}{dt}y(t) &= x(t) * (\frac{d}{dt}h(t)) \end{aligned}$$

ii. (5 points) $\frac{d}{dt}y(t) = (\frac{d}{dt}x(t)) * h(t)$

Solution: Due to the commutative property of convolution,

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned}$$

Now, taking the derivative with respect to t of both sides

$$\frac{d}{dt}y(t) = \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Assuming that the functions are sufficiently smooth, the derivative can be pulled through the integral

$$\begin{aligned}
\frac{d}{dt}y(t) &= \int_{-\infty}^{\infty} \frac{d}{dt}\{h(\tau)x(t-\tau)\}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau \\
\frac{d}{dt}y(t) &= h(t) * (\frac{d}{dt}x(t)) = (\frac{d}{dt}x(t)) * h(t)
\end{aligned}$$

3. **Fourier Series** (30 points).

(a) (15 points) Find the Fourier series of the function: $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq \pi \end{cases}$. To receive full credit, you must simplify your answer, including complex exponentials if possible.

Solution:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = \frac{1}{2\pi}\left[(-x)|_{-\pi}^{0} + x|_{0}^{\pi}\right] = \frac{1}{2\pi}(-\pi+\pi) = 0$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-jnx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,e^{-jnx}dx + \int_{0}^{\pi} e^{-jnx}dx\right]$$
$$= \frac{1}{2\pi}\left[-\frac{\left(e^{-jnx}\right)|_{-\pi}^{0}}{-jn} + \frac{\left(e^{-jnx}\right)|_{0}^{\pi}}{-jn}\right] = \frac{j}{2\pi n}\left[-\left(1-e^{jn\pi}\right) + e^{-jn\pi}-1\right] = \frac{j}{2\pi n}\left[e^{jn\pi} + e^{-jn\pi}-2\right]$$
$$= \frac{j}{\pi n}\left[\frac{e^{jn\pi} + e^{-jn\pi}}{2}-1\right] = \frac{j}{\pi n}\left[\cos n\pi-1\right] = \frac{j}{\pi n}\left[(-1)^n-1\right].$$

If $n = 2k$, then $c_{2k} = 0$. If $n = 2k-1$, then $c_{2k-1} = -\frac{2i}{(2k-1)\pi}$.
Hence, the Fourier series of the function is:

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{2k-1}e^{i(2k-1)x}.$$

(b) (15 points) Let $x(t)$ be a periodic signal whose Fourier series coefficients are:

$$c_k = \begin{cases} 2 & k = 0 \\ j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$$

Use Fourier series properties to answer the following questions:

i. Is $x(t)$ real?
ii. Is $x(t)$ even?
iii. Is $\frac{dx(t)}{dt}$ even?

Solution:

i. We know that if $x(t)$ is real, then $c_k = c_{-k}^*$. We can see that,

$$c_{-k}^* = \begin{cases} 2 & k = 0 \\ -j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$$

Therefore we can see that $c_k \neq c_{-k}^*$. $x(t)$ is not real.

ii. We know that if $x(t)$ is even, then $c_k = c_{-k}$. We can see that,

$c_{-k} = \begin{cases} 2 & k = 0 \\ j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$

Therefore we can see that $c_k = c_{-k}$. $x(t)$ is even.

iii. We know that,

$$x(t) = \sum_{k=\infty}^{\infty} c_k e^{jk\omega_0 t}$$

then,

$$\frac{dx(t)}{dt} = \sum_{k=\infty}^{\infty} c_k jk\omega_0 e^{jk\omega_0 t}$$

From this we can see that, if b_k denotes the Fourier coefficients of $\frac{dx(t)}{dt}$ then $b_k = c_k jk\omega_0$. So,

$b_k = \begin{cases} 0 & k = 0 \\ -j(\frac{1}{2})^{|k|} k\omega_0 & \text{otherwise} \end{cases}$

From this we can see that $\frac{dx(t)}{dt}$ is not even.

Bonus Question (6 points)

Consider two signals, $x(t) = \text{rect}(t - 2.5)$ and $h(t) = \text{rect}(t)$. We convolve $x(2t)$ and $h(2t)$, i.e.,

$$y(t) = x(2t) * h(2t)$$

Using the flip and drag technique, compute the values of:

- $y(0.5)$
- $y(1)$

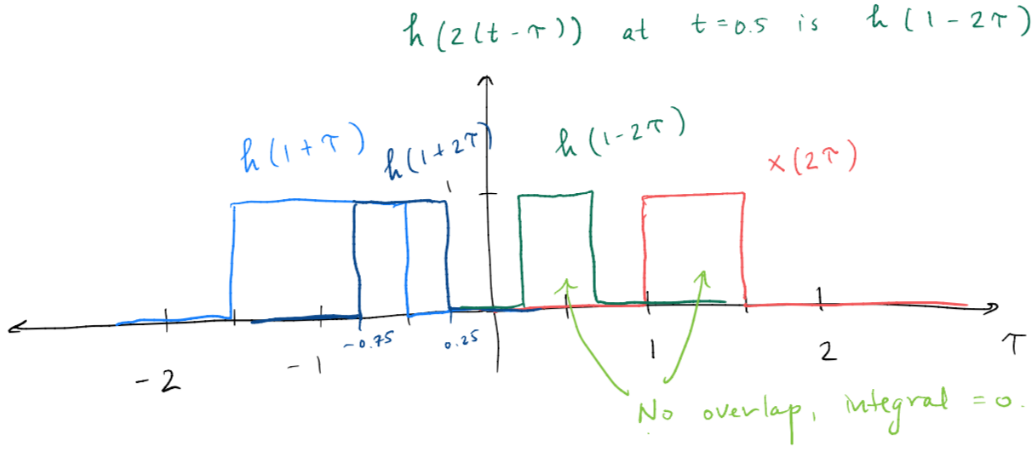
To receive full credit, you must draw a sketch of the flip and drag convolution computation you use to compute $y(0.5)$ and $y(1)$. You will not receive any partial credit for solving this analytically, although you may of course use an analytical answer to check your work. (This is because the purpose of this question is to test your understanding of signal operations and computing convolution.)

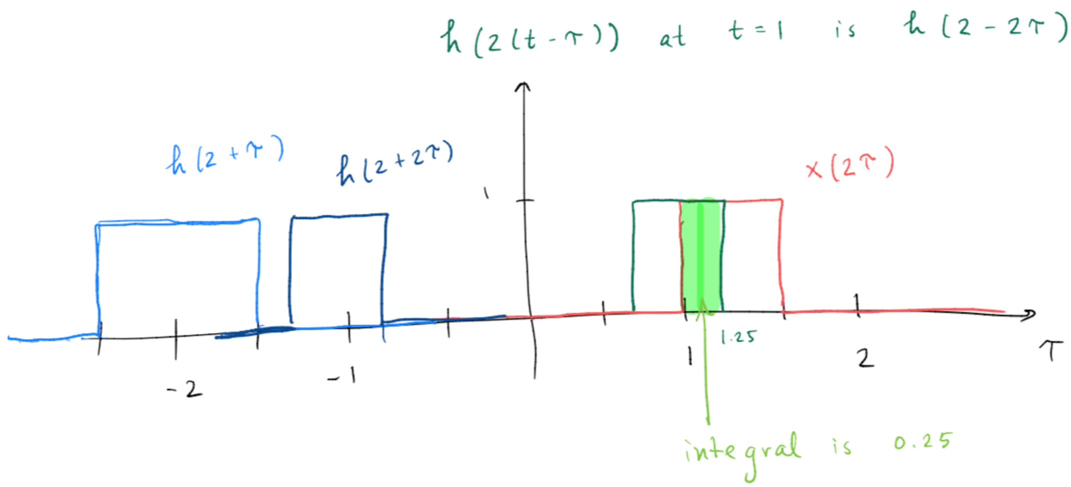
Solution:

The convolution is:

$$y(t) = \int_{-\infty}^{\infty} x(2\tau)h(2(t - \tau))d\tau$$

In the flip and drag, we choose to flip $h(\cdot)$. Then, $x(2\tau)$ is a rect that between $t = 1$ and $t = 1.5$. Below, we illustrate where $h(2(t - \tau))$ is at $t = 0.5$ and $t = 1$. These show that $y(0.5) = 0$ and $y(1) = 0.25$.


h(2(t - τ)) at t = 0.5 is h(1 - 2τ)
h(1 + τ)
h(1 + 2τ)
h(1 - 2τ)
x(2τ)
1
-0.75
0.25
-2
-1
1
2
τ
No overlap, integral = 0.


h(2(t - τ)) at t = 1 is h(2 - 2τ)
h(2 + τ)
h(2 + 2τ)
x(2τ)
1
1.25
-2
-1
1
2
τ
integral is 0.25