ECE102, Fall 2019
Department of Electrical and Computer Engineering
University of California, Los Angeles

Midterm
Prof. J.C. Kao
TAs: W. Feng, J. Lee & S. Wu

UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 13 Nov 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.
There is an extra blank space on page 16 to show your work if you run out of space on any questions.

Name: Shaodian Wang

Signature: SW

ID#: 305111459

Problem 1	____	/ 35
Problem 2	____	/ 20
Problem 3	____	/ 20
Problem 4	____	/ 25
BONUS	____	/ 6 bonus points
Total	____	/ 100 points + 6 bonus points

1. **Signal and System Properties + Convolution** (35 points).

(a) (15 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (5 points) $x(t) = \cos\left(\sqrt{3}t\right) + \sin(-3t)$ is a periodic signal.

False.

The period of $\cos(\sqrt{3}t)$, which is $\frac{2\pi}{\sqrt{3}}$, is not an integer multiple of the period of $\sin(-3t)$, which is $\frac{2\pi}{3}$. $\frac{2\pi}{3}$ is not a integer multiple of $\frac{2\pi}{\sqrt{3}}$ either.

As such, these signals don't have a common period.

ii. (5 points) A signal can be neither energy signal nor power signal.

True. Power signal: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|f(t)\|^2 \, dt < \infty$

Energy signal: $\lim_{T\to\infty} \int_{-T}^{T} \|f(t)\|^2 \, dt < \infty$.

It is possible for a signal to have $\int_{-T}^{T} \|f(t)\|^2 dt \gg 2T$ (evaluated using L'Hopital's rule) such that $\frac{1}{2T}\int_{-T}^{T} \|f(t)\|^2 dt$ is infinity and it is hence neither an energy nor power signal.

iii. (5 points) Let $f(t) * g(t)$ denote the convolution of two signals, $f(t)$ and $g(t)$. Then,

$$f(t)[\delta(t) * g(t)] = [f(t)\delta(t)] * g(t)$$

False. $\delta(t) * g(t)$ evaluates to $g(t)$.

As such, the LHS becomes

$f(t)\,g(t)$.

However, $\int f(t)\delta(t)\,dt$ evaluates to $f(0)$

RHS: $= f(0)g(t) \neq f(t)g(t)$

unless $f(t) = f(0)$ for all t.

(b) (10 points) Determine if the following system is an LTI system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2) \tag{1}$$

Linearity: when input is $a x(t)$,

$$\bar{y}(t) = \frac{a x(t-1)}{t} + a x(t-2)$$

$$= a(y(t)).$$

when input is $x(t) + \tilde{x}(t)$,

$$y(t) = \frac{x(t-1) + \tilde{x}(t-1)}{t} + x(t-2) + \tilde{x}(t-2) = \tilde{y}(t) + y(t)$$

$\Rightarrow$ It is linear.

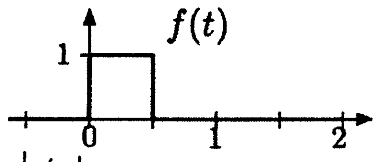
Time invariance: when input is $x(t-T)$,

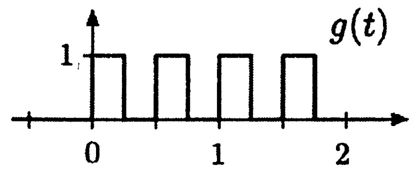
$$\bar{y}(t) = \frac{x(t-T-1)}{t} + x(t-T-2) \neq y(t-T) = \frac{x(t-T-1)}{t-T} + x(t-T-2)$$

$\Rightarrow$ It is not time-invariant.

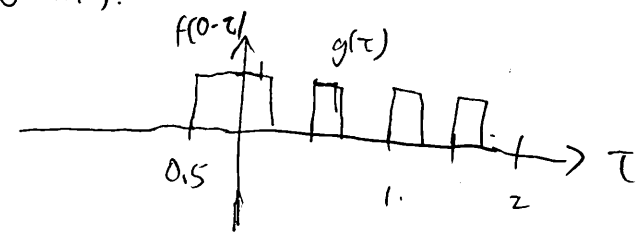
$\Rightarrow$ It is not an LTI system.

(c) (10 points) For signals $f(t)$ and $g(t)$ plotted below, graphically compute the convolution signal $h(t) = f(t) * g(t)$. To receive partial credit, you may show $h(0)$, $h(1/4)$ and $h(5/8)$ in the graph when illustrating the convolution using the "flip and drag" technique.



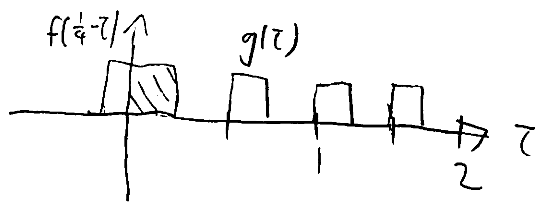


$t=0$ $h(0)$:



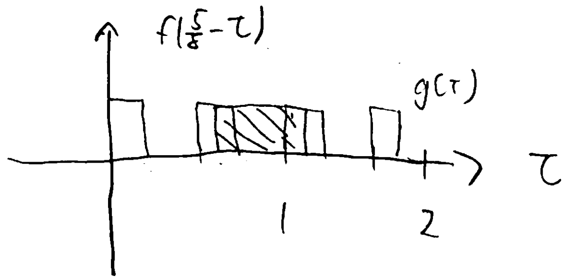
$h(0) = 0$.

$t = 1/4$



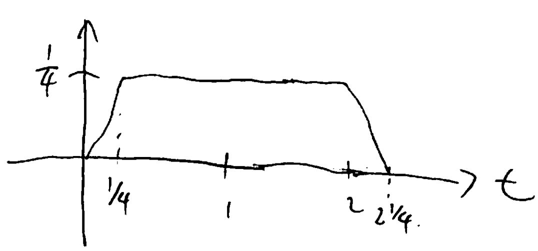
$h(\frac{1}{4}) = \frac{1}{4}$.

$t = 5/8$



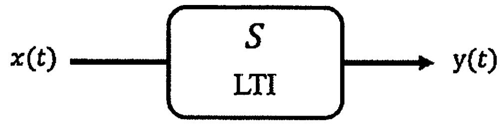
$h(5/8) = \frac{1}{4}$.

$\Rightarrow$ $h(t)$:



2. **LTI Systems** (20 points).

Consider the following LTI system S:



Consider an input signal $x_1(t) = e^{-2t}u(t-2)$. It is given that

$$x_1(t) \xrightarrow{S} y_1(t)$$
$$\frac{dx_1(t)}{dt} \xrightarrow{S} -2y_1(t) + e^{-2t}u(t)$$

(a) (4 points) Show that:

$$\frac{dx_1(t)}{dt} = -2x_1(t) + e^{-2t}\delta(t-2)$$

$$\frac{dx_1(t)}{dt} = \frac{d}{dt}(e^{-2t}u(t-2))$$

$$= -2e^{-2t}u(t-2) + e^{-2t}\delta(t-2)$$

$= -2x_1(t) + e^{-2t}\delta(t-2)$ since $Y_1(t) = e^{-2t}u(t-2)$.

(b) (10 points) Find the impulse response $h(t)$ of S.

Hint: Since we have not provided S, we cannot straightforwardly input an impulse into the system and measure the output. One approach is to solve for $h(t)$ by writing the output of S in terms of a convolution when the input is $\mathrm{d}x_1(t)/\mathrm{d}t$, i.e.,

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} * h(t)$$

$$\frac{dx_1(t)}{dt} * h(t) = \left(-2x_1(t) + e^{-2t}\delta(t-2)\right) * h(t)$$

$$= (-2x_1(t)) * h(t) + (e^{-2t}\delta(t-2)) * h(t)$$

$$= -2y_1(t) + e^{-2t}\delta(t-2) * (h(t))$$

$$\Rightarrow (e^{-2t}\delta(t-2)) * h(t) = e^{-2t}u(t).$$

$$\int_{-\infty}^{\infty} e^{-2\tau}\delta(\tau-2)\, h(t-\tau)\, d\tau = e^{-2t}u(t).$$

$$\Rightarrow e^{-4}\int_{-\infty}^{\infty}\delta(\tau-2)\, h(t-\tau)\, d\tau = e^{-2t}u(t).$$

$$\Rightarrow \int_{-\infty}^{\infty}\delta(\tau-2)\, h(t-\tau)\, d\tau = e^{-2(t-2)}u(t).$$

$$\Rightarrow h(t-2)\int_{-\infty}^{\infty}\delta(\tau-2)\, d\tau = e^{-2(t-2)}u(t).$$

$$\Rightarrow h(t) = e^{-2(t)}u(t+2)$$

(c) (6 points) Consider a new system, S_2, whose impulse response is $h_2(t) = e^{-3t}u(t+3)$. Find this system's output to the following input signal:

$$x_2(t) = \cos\left(\frac{\pi}{4}t\right)\delta(t-1)$$

Output $y_2(t) = h_2(t) * x_2(t)$,

$$= \int_{-\infty}^{\infty} e^{-3(t-\tau)} u(t-\tau+3) \cos(\pi/4\,\tau)\,\delta(\tau-1)\,d\tau$$

$$= \cos \pi/4 \; e^{-3(t-1)} u(t+2) \int_{-\infty}^{\infty} \delta(\tau-1)\,d\tau$$

$$= \cos \pi/4 \; e^{-3(t-1)} u(t+2)$$

3. **Fourier Series** (20 points).

(a) (10 points) Let the Fourier Series coefficients of $f(t)$ be denoted f_k, and the Fourier Series coefficients of $g(t)$ denoted g_k. Let T_o be the period of $f(t)$. If $g(t) = f(a(t-b))$, where $a > 0$, show that

$$g_k = e^{-j2\pi \frac{ab}{T_o}k} f_k.$$

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{+j\frac{2\pi}{T_0}kt}.$$

$$g(t) = f(at-ab) = \sum_{k=-\infty}^{\infty} f_k e^{+j\frac{2\pi}{T_0}k(at-ab)}$$

$$= \sum_{k=-\infty}^{\infty} f_k e^{-j\frac{2\pi}{T_0}abk} e^{j\frac{2\pi}{T_0}akt}$$

Since $g(t) = f(a(t-b))$, period T_1 of $g(t) = T_0/a$.

$$\Rightarrow \quad g(t) = \sum_{k=-\infty}^{\infty} f_k e^{-j\frac{2\pi}{T_0}abk} e^{j\frac{2\pi}{T_1}kt}$$

$$\Rightarrow \quad g_k = f_k e^{-j\frac{2\pi}{T_0}abk}.$$

$T_1 = \frac{T_0}{m}$

(b) (10 points) Let the Fourier Series coefficients of $x(t)$ and $y(t)$ be x_k and y_k respectively, with respective periods T_1 and T_2. We define $f(t) = \alpha_1 x(t) + \alpha_2 y(t)$ with non-zero α_1, α_2, with period $T_o = m_1 T_1 = m_2 T_2$. What are the Fourier Series coefficients f_k in terms of x_k and y_k?

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T_1}kt}$$

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j\frac{2\pi}{T_2}kt}.$$

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{j\frac{2\pi}{T_0}kt}.$$

$$f(t) = \sum_{k=-\infty}^{\infty} \alpha_1 X_k e^{j\frac{2\pi}{T_0}m_1 kt} + \alpha_2 Y_k e^{j\frac{2\pi}{T_0}m_2 kt} = \sum_{k=-\infty}^{\infty} f_k e^{j\frac{2\pi}{T_0}kt}$$

By comparing coefficients,

$\Rightarrow f_k = \alpha_1 X_{k/m_1} + \alpha_2 Y_{k/m_2}$ if k is divisible by m_1 & m_2,

$f_k = \alpha_1 X_{k/m_1}$ if k is divisible by m_1,

$f_k = \alpha_2 Y_{k/m_2}$ if k is divisible by m_2,

$f_k = 0$ otherwise.

4. **Fourier Transform** (25 points).
Consider the signal

$$x(t) = \text{sinc}(2t)$$

and let the Fourier transform of $x(t)$ be denoted $X(j\omega)$. We are interested in calculating the area under the curve of $x(t)$.

(a) (10 points) Prove that the following relationship holds.

$$\int_{-\infty}^{\infty} x(t)dt = X(j\omega)|_{\omega=0}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

At $\omega = 0$, this becomes

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t) e^{0} dt$$

$$= \int_{-\infty}^{\infty} x(t) dt$$

(b) (5 points) Use the result of part (a) to calculate:

$$\int_{-\infty}^{\infty} x(t)dt$$

for $x(t) = \text{sinc}(2t)$.

$$\text{sinc}(t) \Longleftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\text{sinc}(2t) \Longleftrightarrow \frac{1}{2}\,\text{rect}\left(\frac{\omega}{4\pi}\right)$$

Since $\int_{-\infty}^{\infty} \text{sinc}(2t)\,dt = \frac{1}{2}\,\text{rect}\left(\frac{\omega}{4\pi}\right)\Big|_{\omega=0}$,

$$\int_{-\infty}^{\infty} \text{sinc}(2t)dt = \frac{1}{2}.$$

(c) (5 points) Consider the following system:

$$y(t) = e^{-j\omega_0 t}x(t)$$

Let $x(t) = \text{sinc}(2t)$ and consider only $\omega_0 > 0$. Are there any values of ω_0 for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of ω_0 does this hold for?

$$y(t) \Longleftrightarrow \frac{1}{2}\text{rect}\left(\frac{w+w_0}{4\pi}\right)$$

$$\int_{-\infty}^{\infty} y(t)\,dt = \frac{1}{2}\text{rect}\left(\frac{w+w_0}{4\pi}\right)\Bigg|_{w=0}.$$

$$= \frac{1}{2}\text{rect}\left(\frac{w_0}{4\pi}\right).$$

Yes. the values are $w_0 > 2\pi$ or $w_0 < -2\pi$.

(d) (5 points) Consider the following system:

$$y(t) = x(t) + \alpha \mathrm{rect}(t)$$

Let $x(t) = \mathrm{sinc}(2t)$. Are there any values of α for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of α does this hold for?

$$y(t) \Longleftrightarrow \frac{1}{2}\mathrm{rect}\left(\frac{\omega}{4\pi}\right) + \alpha \,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right).$$

$$\int_{-\infty}^{\infty} y(t)\,dt = \left.\frac{1}{2}\mathrm{rect}\left(\frac{\omega}{4\pi}\right) + \alpha\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)\right|_{\omega=0}$$

$$= \frac{1}{2} + \alpha\,\mathrm{sinc}(0).$$

$$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}.$$

$$\lim_{\omega \to 0} \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} = \frac{\frac{1}{2}\cos(\omega/2)}{1/2} \quad \text{(L'Hopital's rule).}$$

$$= \frac{1/2}{1/2} = 1.$$

$$\Rightarrow \int_{-\infty}^{\infty} y(t)\,dt = \frac{1}{2} + \alpha(1).$$

$= 0$ when $\alpha = -\frac{1}{2}$.

Bonus (6 points) Suppose $\cancel{x(t)}$ $f(t)$ $= \cos(\omega_o t)$ is an eigenfunction of an LTI system S for any ω_o, and S cannot be defined as $S[x(t)] = ax(t)$ for some constant a. Is the system S causal? Justify your answer.

Causal: only takes in input from $t < T$

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

is an eigenfunction =

input signal $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} X(j\omega) \cos\omega t\, d\omega + j\int_{-\infty}^{\infty} X(j\omega)\sin\omega t\, d\omega\right)$$

This is an extra piece of paper to show your work. If you use this space for a question, for that question, please write "Refer to page 16."