ECE102, Fall 2019 **Midterm**

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UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 13 Nov 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.
There is an extra blank space on page 16 to show your work if you run out of space on any questions.

Name: [redacted]

Signature: [redacted]

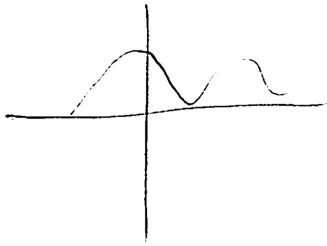
ID#: [redacted]

Problem 1	____	/ 35
Problem 2	____	/ 20
Problem 3	____	/ 20
Problem 4	____	/ 25
BONUS	____	/ 6 bonus points
Total	____	/ 100 points + 6 bonus points

1. **Signal and System Properties + Convolution** (35 points).

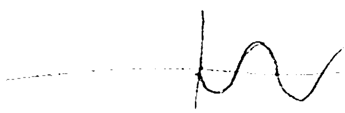
(a) (15 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (5 points) $x(t) = \cos\left(\sqrt{3}t\right) + \sin(-3t)$ is a periodic signal.



$\cos(\sqrt{3}t)$ is periodic because it is a sinusoid

Period $T_1 = \frac{2\pi}{\sqrt{3}}$



$\sin(-3t)$ is periodic because it is a sinusoid

Period $T_2 = \frac{2\pi}{-3}$

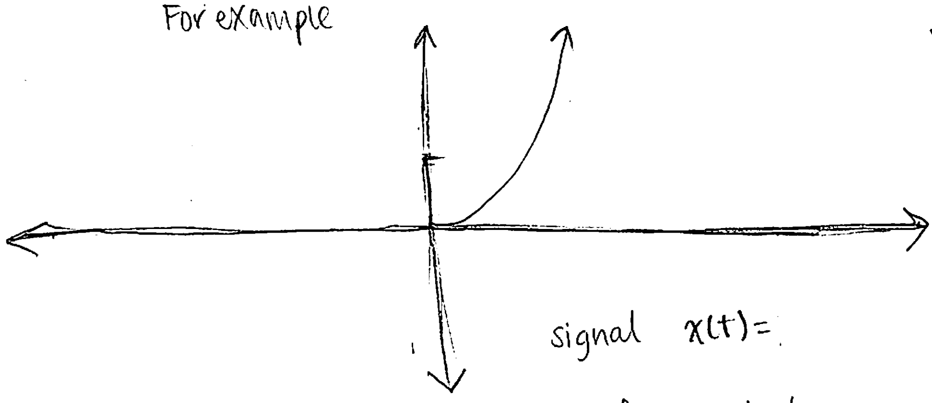
$T_0 = k_1 T_1 = k_2 T_2$

$= k_1 \frac{2\pi}{\sqrt{3}} = k_2 \frac{2\pi}{-3}$

There does not exist a T_0 such that $k_1 T_1 = k_2 T_2 = T_0$ where k_1 and k_2 are integers, so this is not a periodic signal

ii. (5 points) A signal can be neither energy signal nor power signal.

For example



Yes. One instance is e^t

signal $x(t) =$

$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$= \int_{-\infty}^{\infty} 0^2 dt$

$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1^2 dt = \lim_{T\to\infty} \frac{1}{2T}(T+T)$

$= \lim_{T\to\infty} \frac{1}{2T} \left[t\right]_{-T}^{T}$ $= \lim$

iii. (5 points) Let $f(t)*g(t)$ denote the convolution of two signals, $f(t)$ and $g(t)$. Then,

$$f(t)[\delta(t)*g(t)] = [f(t)\delta(t)]*g(t)$$

$$f(t)[\delta(t)*g(t)] = f(t)\delta(t) * f(t)g(t)$$

$$= f_0\,\delta(t) * f(t)g(t)$$

$$= f_0 * f(t)g(t)$$

$$[f(t)\delta(t)]*g(t) = f(t)*g(t)\cdot\delta(t)*g(t)$$

this is false

$$f(t)\cdot g(t) = f(t)*g(t) \ ?$$

Counterexample:

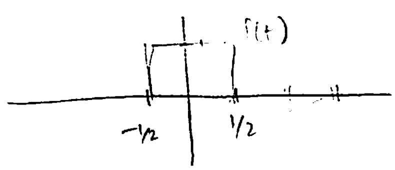
let $f(t) = \text{rect}(t)$

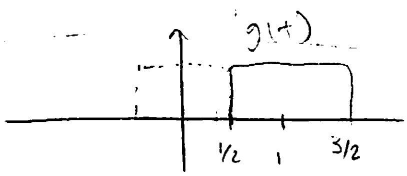
$g(t) = \text{rect}(t-2)$

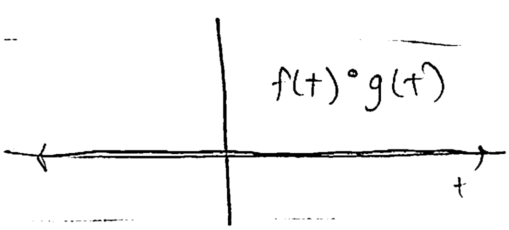
$f(t)\cdot g(t) = 0$

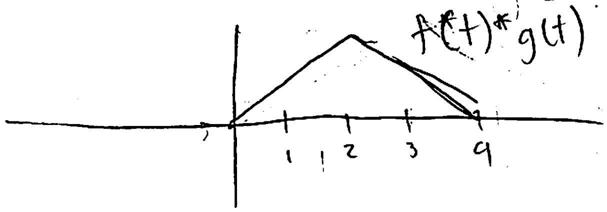
$f(t)*g(t) \neq 0$

using flip & drag









(b) (10 points) Determine if the following system is an LTI system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2) \tag{1}$$

Linearity:

Show $S[ax_1(t) + bx_2(t)] = aS[x_1(t)] + bS[x_2(t)]$

LHS: $\dfrac{ax_1(t-1) + bx_2(t-1)}{t} + ax_1(t-2) + bx_2(t-2)$

RHS: $a\left(\dfrac{x_1(t-1)}{t} + x_1(t-2)\right) + b\left(\dfrac{x_2(t-1)}{t} + x_2(t-2)\right)$

$= \dfrac{ax_1(t-1)}{t} + ax_1(t-2) + \dfrac{bx_2(t-1)}{t} + bx_2(t-2)$

$= \dfrac{ax_1(t-1) + bx_2(t-1)}{t} + ax_1(t-2) + bx_2(t-2)$

RHS = LHS, $\therefore$ System is linear

Time Invariant:

Show $y(t) = S[x(t)]$

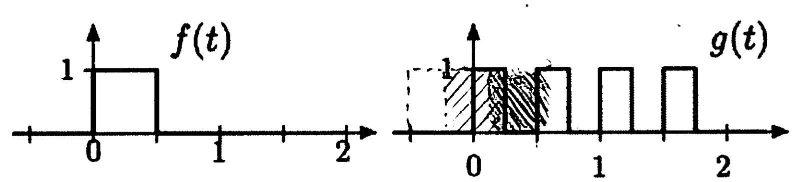
delay input: output $= \dfrac{x(t-1-\alpha)}{t} + x(t-2-\alpha)$

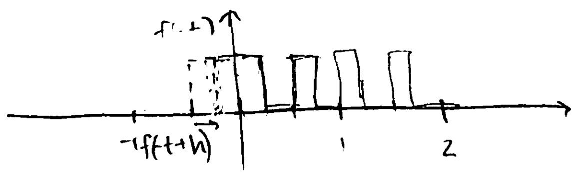
delay output: $y(t-\alpha) = \dfrac{x(t-1-\alpha)}{t-\alpha} + x(t-2-\alpha)$

output $\neq y(t-\alpha)$

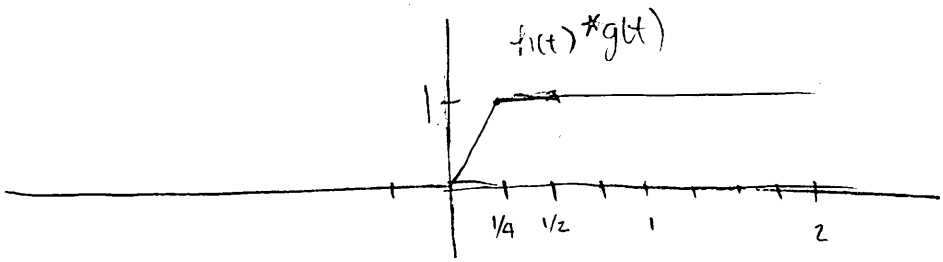
$\therefore$ not time invariant system

(c) (10 points) For signals $f(t)$ and $g(t)$ plotted below, graphically compute the convolution signal $h(t) = f(t) * g(t)$. To receive partial credit, you may show $h(0)$, $h(1/4)$ and $h(5/8)$ in the graph when illustrating the convolution using the "flip and drag" technique.





$5/8 - 1/2 = 1/8$



$h(0) = 0$, no overlap

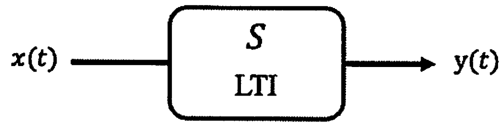
$h(1/4)$ partial overlap

$= 1 \cdot 1$

$h(5/8) = 1 \cdot 1$

2. **LTI Systems** (20 points).

Consider the following LTI system S:



Consider an input signal $x_1(t) = e^{-2t}u(t-2)$. It is given that

$$x_1(t) \xrightarrow{\quad S \quad} y_1(t)$$
$$\frac{dx_1(t)}{dt} \xrightarrow{\quad S \quad} -2y_1(t) + e^{-2t}u(t)$$

(a) (4 points) Show that:

$$\frac{dx_1(t)}{dt} = -2x_1(t) + e^{-2t}\delta(t-2)$$

$$\cancel{x_1(t) = e^{-2t}u(t-2) \to y_1(t)}$$
$$\cancel{\frac{dx_1(t)}{dt} = \frac{d}{dt}\left(e^{-2t}u(t-2)\right) \to -2y_1(t) + e^{-2t}u(t)}$$

$$\frac{dx_1(t)}{dt} = \frac{d}{dt}\left(e^{-2t}u(t-2)\right) \quad \text{chain rule} + \tfrac{d}{dt}u(t-2) = \delta(t-2)$$
$$= -2e^{-2t}\cdot u(t-2) + \delta(t-2)e^{-2t}$$
$$= -2x_1(t) + e^{-2t}\delta(t-2)$$

(b) (10 points) Find the impulse response $h(t)$ of S.

Hint: Since we have not provided S, we cannot straightforwardly input an impulse into the system and measure the output. One approach is to solve for $h(t)$ by writing the output of S in terms of a convolution when the input is $\mathrm{d}x_1(t)/\mathrm{d}t$, i.e.,

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} * h(t)$$

$$\frac{dx_1(t)}{dt} * h(t) = -2y_1(t) + e^{-2t}u(t)$$

$$(-2x_1(t) + e^{-2t}\delta(t-2)) * h(t) = -2y_1(t) + e^{-2t}u(t)$$

$$-2x_1(t) * h(t) + e^{-2t}\delta(t-2) * h(t) = -2y_1(t) + e^{-2t}u(t)$$

$$-2x_1(t) * h(t) + e^{-2t}h(t-2) = -2y_1(t) + e^{-2t}u(t)$$

$$-2x_1(t) * h(t) = -2y_1(t)$$

$$e^{-2t}h(t-2) = e^{-2t}u(t)$$

$$h(1) = u(3)$$

$$h(t-2) = u(t)$$

$$\boxed{h(t) = u(t+2)}$$

(c) (6 points) Consider a new system, S_2, whose impulse response is $h_2(t) = e^{-3t}u(t+3)$. Find this system's output to the following input signal:

$$x_2(t) = \cos\left(\frac{\pi}{4}t\right)\delta(t-1)$$

$x_2(t) * h_2(t) = y_2(t)$

$\cos(\frac{\pi}{4}t)\,\delta(t-1) * e^{-3t}u(t+3) = y_2(t)$

$y_2(t) = \cos(\frac{\pi}{4}t)\,\delta(t-1) * e^{-3t}u(t+3)$

$= \cos(\frac{\pi}{4})\,\delta(t-1) * e^{-3t}u(t+3)$

$= \cos(\frac{\pi}{4})\,e^{-3(t-1)}u(t-1+3)$

$= \cos(\frac{\pi}{4})\,e^{-3t+3}u(t+2)$

3. **Fourier Series** (20 points).

(a) (10 points) Let the Fourier Series coefficients of $f(t)$ be denoted f_k, and the Fourier Series coefficients of $g(t)$ denoted g_k. Let T_o be the period of $f(t)$. If $g(t) = f(a(t-b))$, where $a > 0$, show that

$$g_k = e^{-j2\pi \frac{ab}{T_o} k} f_k.$$

$f(t)$ f_K T_0

$g(t)$ g_K

$$g(t) = f(a(t-b))$$

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} f_k e^{jk\frac{2\pi}{T_0}t}$$

$$g(t) = \sum_{k=-\infty}^{\infty} g_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} g_k e^{jk\frac{2\pi}{T_1}t}$$

$$g(t) = f(a(t-b))$$

$$= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0(a(t-b))}$$

$\omega_0 = \frac{2\pi}{T_0}$

$$= \sum_{k=-\infty}^{\infty} f_k e^{jk\frac{2\pi}{T_0}at} \cdot e^{-jk\frac{2\pi}{T_0}ab}$$

$$= \sum_{k=-\infty}^{\infty} \left(f_k e^{-j2\pi\frac{ab}{T_0}k}\right) e^{jk\frac{2\pi}{T_0}at}$$

let $g(t)$'s period $T_1 = \frac{T_0}{a}$

$\omega_1 = \frac{2\pi a}{T_0}$

$$\therefore g_k = e^{j2\pi\frac{ab}{T_0}k} f_k$$

(b) (10 points) Let the Fourier Series coefficients of $x(t)$ and $y(t)$ be x_k and y_k respectively, with respective periods T_1 and T_2. We define $f(t) = \alpha_1 x(t) + \alpha_2 y(t)$ with non-zero α_1, α_2, with period $T_o = m_1 T_1 = m_2 T_2$. What are the Fourier Series coefficients f_k in terms of x_k and y_k?

$\omega_0 = \frac{2\pi}{m_1 T_1}$

$x(t) \cdot x_k \quad T_1 \qquad y(t) \quad y_k \quad T_2$

$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} f_k e^{jk\frac{2\pi}{T_0}t}$

$f(t) = \alpha_1 x(t) + \alpha_2 y(t)$

$= \alpha_1 \sum_{k=-\infty}^{\infty} x_k e^{jk\frac{2\pi}{T_1}t} + \alpha_2 \sum_{k=-\infty}^{\infty} y_k e^{jk\frac{2\pi}{T_2}t}$

$= \sum_{k=-\infty}^{\infty} \alpha_1 x_k e^{jk\frac{2\pi}{T_1}t} + \alpha_2 y_k e^{jk\frac{2\pi}{T_2}t}$

$= \sum_{k=-\infty}^{\infty} \alpha_1 x_k e^{jk\frac{2\pi m_1 t}{T_0}} + \alpha_2 y_k e^{jk\frac{2\pi m_2}{T_0}t}$

$T_0 = m_1 T_1$

$T_1 = \frac{T_0}{m_1} \qquad w_1 = 2\pi \cdot \frac{m_1}{T_0}$

$T_2 = \frac{T_0}{m_2}$

$= \sum_{k=-\infty}^{\infty} \alpha_1 x_k e^{jk\frac{2\pi t}{T_0}} \cdot e^{\frac{m_1}{T_0}} + \alpha_2 y_k e^{jk\frac{2\pi}{T_0}t} \cdot e^{\frac{m_2}{T_0}}$

$= \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T_0}t} \left(\alpha_1 x_k e^{\frac{m_1}{T_0}} + \alpha_2 y_k e^{\frac{m_2}{T_0}} \right)$

$\therefore f_k = \alpha_1 x_k e^{\frac{m_1}{T_0}} + \alpha_2 y_k e^{\frac{m_2}{T_0}}$

4. **Fourier Transform** (25 points).
Consider the signal

$$x(t) = \text{sinc}(2t)$$

and let the Fourier transform of $x(t)$ be denoted $X(j\omega)$. We are interested in calculating the area under the curve of $x(t)$.

(a) (10 points) Prove that the following relationship holds.

$$\int_{-\infty}^{\infty} x(t)dt = X(j\omega)|_{\omega=0}$$

Definition of Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t) e^{-j\cdot 0\cdot t} dt$$

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t) dt$$

(b) (5 points) Use the result of part (a) to calculate:

$$\int_{-\infty}^{\infty} x(t)dt$$

for $x(t) = \text{sinc}(2t)$.

$$X(j\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)\,dt$$

$$x(t) = \text{sinc}(2t)$$

$$x(t) \longleftrightarrow X(j\omega)$$

$$\text{sinc}(1t) \longleftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\text{sinc}(2t) \longleftrightarrow \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

$$\therefore X(j\omega) = \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right)$$

$$X(j\omega)\big|_{\omega=0} = \frac{1}{2}\text{rect}\left(\frac{0}{4\pi}\right)$$

$$= \frac{1}{2}\text{rect}(0)$$

$$= \frac{1}{2}\cdot 1 = \frac{1}{2}$$

(c) (5 points) Consider the following system:

$$y(t) = e^{-j\omega_0 t}x(t)$$

Let $x(t) = \text{sinc}(2t)$ and consider only $\omega_0 > 0$. Are there any values of ω_0 for which

$$Y(j\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of ω_0 does this hold for?

$$y(t) = e^{-j\omega_0 t}x(t)$$

$$y(t) \longleftrightarrow Y(j\omega)$$

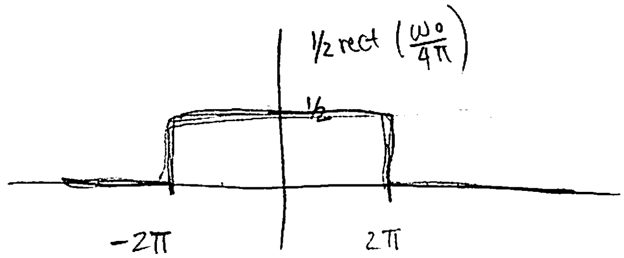
$$x(t)e^{-j\omega_0 t} \longleftrightarrow X(j(\omega-\omega_0))$$

$$Y(j\omega) = X(j(\omega-\omega_0))$$

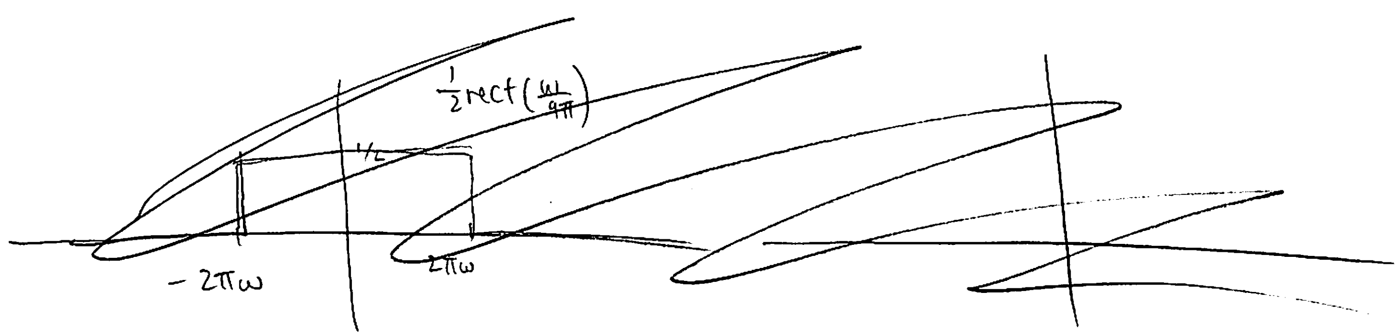
$$= \frac{1}{2}\text{rect}\left(\frac{\omega-\omega_0}{4\pi}\right)$$

$$Y(j\omega)\big|_{\omega=0} = \frac{1}{2}\text{rect}\left(\frac{-\omega_0}{4\pi}\right)$$

$$0 = \frac{1}{2}\text{rect}\left(\frac{-\omega_0}{4\pi}\right)$$



$$\boxed{\omega_0 > 2\pi}$$



(d) (5 points) Consider the following system:

$$y(t) = x(t) + \alpha \text{rect}(t)$$

Let $x(t) = \text{sinc}(2t)$. Are there any values of α for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of α does this hold for?

$y(t) \longleftrightarrow Y(j\omega)$

$x(t) + \alpha \,\text{rect}(t) \longleftrightarrow X(j\omega) + \alpha \,\text{sinc}\left(\frac{\omega}{2\pi}\right)$

$Y(j\omega) = X(j\omega) + \alpha \,\text{sinc}\left(\frac{\omega}{2\pi}\right)$

$Y(j\omega) = \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right) + \alpha \,\text{sinc}\left(\frac{\omega}{2\pi}\right)$

$Y(j\omega)\Big|_{\omega=0} = \frac{1}{2}\text{rect}\left(\frac{\omega}{4\pi}\right) + \alpha \,\text{sinc}\left(\frac{\omega}{2\pi}\right)$

$Y(j\omega)\Big|_{\omega=0} = \frac{1}{2}\text{rect}(0) + \alpha \,\text{sinc}(0)$

$0 = \frac{1}{2}\text{rect}(0) + \alpha \,\text{sinc}(0)$

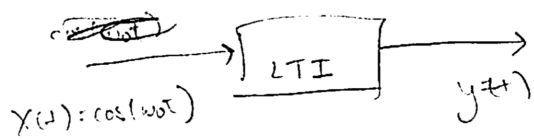
$\alpha \,\text{sinc}(0) = -\frac{1}{2}$

$\alpha = -\frac{1}{2}$

Bonus (6 points) Suppose $x(t) = \cos(\omega_o t)$ is an eigenfunction of an LTI system S for any ω_o, and S cannot be defined as $S[x(t)] = ax(t)$ for some constant a. Is the system S causal? Justify your answer.

$S[x(t)] \neq ax(t)$

cos



causal ~ depends on only past/present time

This is an extra piece of paper to show your work. If you use this space for a question, for that question, please write "Refer to page 16."