ECE102, Fall 2020 **Final Exam**

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UCLA True Bruin academic integrity principles apply.
Open: Notes, Book.
Closed: Internet, except to use Piazza and CCLE.
3:00-6:00pm.
Wednesday, 16 Dec 2020.

State your assumptions and reasoning.
No credit without reasoning.

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Signature: Ziyi Yang

ID#: 405064363

Problem 1	____	/ 50
Problem 2	____	/ 50
Problem 3	____	/ 54
Problem 4	____	/ 46
BONUS	____	/ 15 bonus points
Total	____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (50 points)

   (a) (16 points) Consider an LTI system whose response to the signal $x_1(t)$ in Figure (1a) is the signal $y_1(t)$ illustrated in Figure (1b).

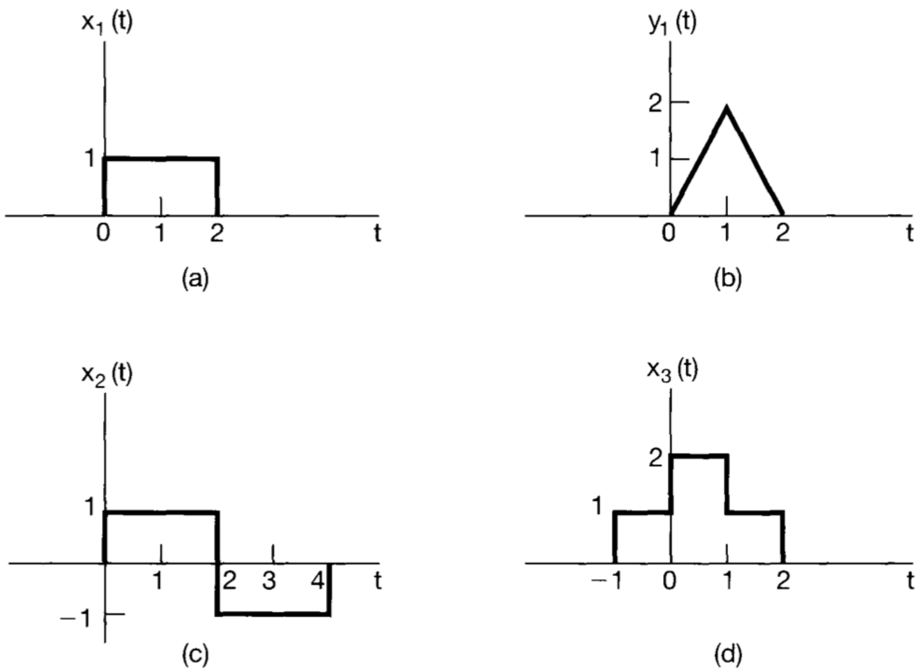


Figure 1: Input-output relationship

i. (8 points) Determine and sketch the response of the system to the input $x_2(t)$ shown in Figure (1c).

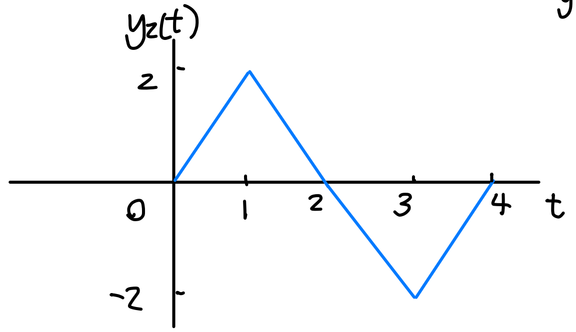
$x_2(t) = x_1(t) - x_1(t-2)$

since we pass x_1, x_2 into a LTI system S, ← linearity

$y_2(t) = S(x_1(t) - x_1(t-2)) = S(x_1(t)) - S(x_1(t-2))$

$= y_1(t) - y_1(t-2)$

← time-invariance



ii. (8 points) Determine and sketch the response of the system to the input $x_3(t)$ shown in Figure (1d).

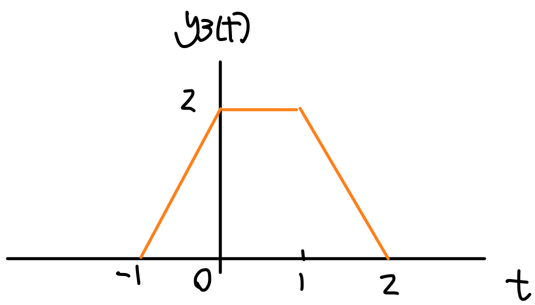
$x_3(t) = x_1(t) + x_1(t+1)$

since the system S is LTI and $S(x_1(t)) = y_1(t)$,

$S(x_3(t)) = S(x_1(t) + x_1(t+1))$

$= S(x_1(t)) + S(x_1(t+1))$ by linearity

$= y_1(t) + y_1(t+1)$ by time invariance



(b) (24 points) For each statement below, determine whether it is true or false. You must justify your answer to receive full credit.

i. (8 points) If $T_1(x)$ and $T_2(x)$ are LTI systems, then $T_2(T_1(x))$ must be LTI.

<u>True</u>

since $T_1(x)$, $T_2(x)$ are LTI systems

there exist impulse response functions $h_1(t)$, $h_2(t)$ such that

$y_1(t) = h_1(t) * x_1(t)$ and $y_2(t) = h_2(t) * x_2(t)$

$$x(t) \rightarrow \boxed{T_1} \xrightarrow[z(t)]{} \boxed{T_2} \rightarrow y(t)$$

$$T_2(T_1(x)) = h_2(t) * z(t) = h_2(t) * (h_1(t) * x(t))$$

$$= \underbrace{(h_2(t) * h_1(t))}_{h_{eq}(t)} * x(t) \quad \text{by associativity of convolution}$$

since we could see $h_{eq} = h_2 * h_1$ as an equivalent impulse response,

$T_2(T_1(x))$ must be LTI system

ii. (8 points) If $T_1(x)$ and $T_2(x)$ are nonlinear systems, then $T_2(T_1(x))$ must be nonlinear.

<u>False</u>

let $T_1(x) = x^2$, $T_2(x) = \sqrt{x}$, both T_1, T_2 are non-linear by inspecting the order of x

then $T_2(T_1(x)) = \sqrt{x^2} = x$ which is linear

$$T_2(T_1(ax + b\tilde{x})) = ax + b\tilde{x}$$

$$= a\,T_2(T_1(x)) + b\,T_2(T_1(\tilde{x})) \Rightarrow \text{linear}$$

So the statement is false

iii. (8 points) Consider a time-invariant system with input $x(t)$ and output $y(t)$. If $x(t)$ is periodic with period T, then $y(t)$ is also periodic with period T.

<u>True</u>

Since $x(t)$ is periodic with period T,

$$x(t+T) = x(t) \text{ for all } t$$

denote the system by S, then that means

$$S(x(t+T)) = S(x(t)) \text{ for all } t$$

$$y(t+T) = y(t) \text{ for all } t \Rightarrow y(t) \text{ is also periodic with period } T$$

by time-invariance

the statement is True

(c) (10 points) The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is:

$$y(t) = x_1(t) * x_2(t) \tag{1}$$

where,

$$X_1(j\omega) = 0 \text{ for } |\omega| > 1000\pi$$
$$X_2(j\omega) = 0 \text{ for } |\omega| > 2000\pi$$

Next, $y(t)$ is sampled via an impulse train to obtain

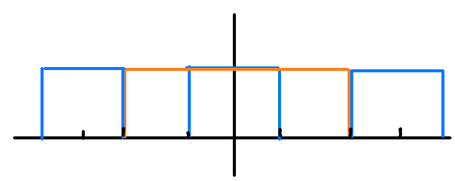
$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT)\delta(t - nT)$$

Specify the range of values for the sampling period T which ensures that $y(t)$ is recoverable from $y_p(t)$.

the bandwidth of $x_1(t)$ is 1000π, of $x_2(t)$ is 2000π

since $y(t) = x_1(t) * x_2(t)$,

$y(t)$ has a bandwidth of 3000π



for illustration, I portray x_1, x_2 as rect functions and perform a flip and drag

we can see there is no overlap beyond $\omega = \pm 3000\pi$, so the new bandwidth is 3000π

$2\pi B = 3000\pi$

$B = 1500$

Nyquist interval $T = \frac{1}{2B} = \frac{1}{3000}$

thus to ensure $y(t)$ is recoverable, $T < \frac{1}{3000}$

2. **Fourier transform, Frequency response and Output of LTI systems** (50 points)

(a) (30 points) Consider an LTI system with the impulse response:

$$h(t) = \left(\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right) 2j\sin(20t)$$

i. (5 points) Show that

$$\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t} = \frac{15}{\pi} Sa(5t)Sa(15t)$$

where $Sa(t) = \frac{sin(t)}{t}$.

$$\text{LHS} = \frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}$$

$$= \frac{\cancel{\pi}}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{\pi}} \frac{\sin(5t)}{5t} \cdot \frac{15}{\pi}\frac{\sin(15t)}{15t}$$

$$= \frac{15}{\pi} Sa(5t) Sa(15t)$$

$$Sa(t) = \frac{\sin(t)}{t}$$

$$Sa(rt) = \frac{\sin(rt)}{rt}$$

ii. (15 points) Compute the Fourier transform of

$$\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}$$

You must indicate which properties and pairs you are using to arrive at your answer. You must show all steps to receive credit.

Hint: You may find the following Fourier Transform useful.

$$\frac{B}{2\pi}Sa\left(\frac{Bt}{2}\right) \Longleftrightarrow \text{rect}\left(\frac{\omega}{B}\right) \qquad Sa\left(\frac{Bt}{2}\right) \Longleftrightarrow \frac{2\pi}{B}\,\text{rect}\left(\frac{\omega}{B}\right)$$

in (i) we've shown $\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t} = \frac{15}{\pi}Sa(5t)Sa(15t)$

$$Sa(5t) \Longleftrightarrow \frac{2\pi}{10}\,\text{rect}\left(\frac{\omega}{10}\right) = \frac{\pi}{5}\,\text{rect}\left(\frac{\omega}{10}\right)$$

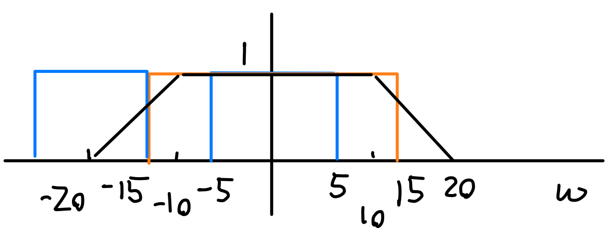
$$Sa(15t) \Longleftrightarrow \frac{2\pi}{30}\,\text{rect}\left(\frac{\omega}{30}\right) = \frac{\pi}{15}\,\text{rect}\left(\frac{\omega}{30}\right)$$

by convolution theorem and duality, $\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}\mathcal{F}[f_1(t)] * \mathcal{F}[f_2(t)]$

$$\mathcal{F}[Sa(5t)Sa(15t)] = \frac{1}{2\pi}\mathcal{F}[Sa(5t)] * \mathcal{F}[Sa(15t)]$$

perform flip and drag

$$= \frac{1}{2\pi}\cdot\frac{\pi^2}{5\cdot 15}\underbrace{\text{rect}\left(\frac{\omega}{10}\right) * \text{rect}\left(\frac{\omega}{30}\right)}$$



$$= \begin{cases} 0 & \text{for } \omega > 20 \text{ or } \omega < -20 \\ \omega + 20 & \text{for } -20 \le \omega \le -10 \\ 10 & \text{for } -10 < \omega < 10 \\ -\omega + 20 & \text{for } 10 \le \omega \le 20 \end{cases}$$

$$\Rightarrow \mathcal{F}\left[\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right] = \frac{15}{\pi}\mathcal{F}[Sa(5t)Sa(15t)]$$

$$= \frac{\cancel{15}}{\cancel{\pi}}\cdot\frac{1}{2\cancel{\pi}}\frac{\cancel{\pi^2}}{5\cdot\cancel{15}}\,\text{rect}\left(\frac{\omega}{10}\right) * \text{rect}\left(\frac{\omega}{30}\right)$$

$$= \begin{cases} 0 & \text{for } |\omega| > 20 \\ 0.1\omega + 2 & \text{for } -20 \le \omega \le -10 \\ 1 & \text{for } |\omega| < 10 \\ -0.1\omega + 2 & \text{for } 10 \le \omega \le 20 \end{cases}$$

iii. (10 points) Use the properties of Fourier transform to determine the frequency response $H(j\omega) = \mathcal{F}[h(t)]$ and plot it on graph on the next page. You must indicate which property you are using to arrive at your answer. You must show all steps to receive credit. For your convenience, recall that $h(t)$ is:

$$h(t) = \underbrace{\left(\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(15t)}{\pi t}\right)}_{f(t)} 2j\sin(20t)$$

$$2j\sin(20t) = 2j \cdot \frac{e^{j20t} - e^{-j20t}}{2j} = e^{j20t} - e^{-j20t}$$

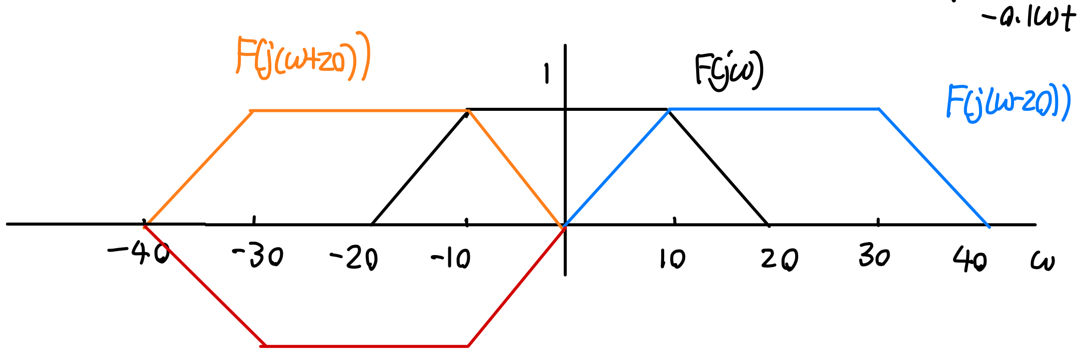
from (ii) we've calculated $F(j\omega) = \mathcal{F}[f(t)]$

$$h(t) = f(t)\left(e^{j20t} - e^{-j20t}\right) = f(t)e^{j20t} - f(t)e^{-j20t}$$

by time-shift property and duality,

$$f(t)\, e^{j\omega_0 t} \Leftrightarrow F(j(\omega - \omega_0))$$

$$\Rightarrow H(j\omega) = \mathcal{F}[h(t)] = F(j(\omega-20)) - F(j(\omega+20)) = \begin{cases} -0.1\omega + 4 & \text{for } -40 \le \omega \le 30 \\ -1 & \text{for } -30 < \omega < -10 \\ 0.1\omega & \text{for } -10 \le \omega \le 10 \\ 1 & \text{for } 10 < \omega < 30 \\ -0.1\omega + 4 & \text{for } 30 \le \omega \le 40 \end{cases}$$



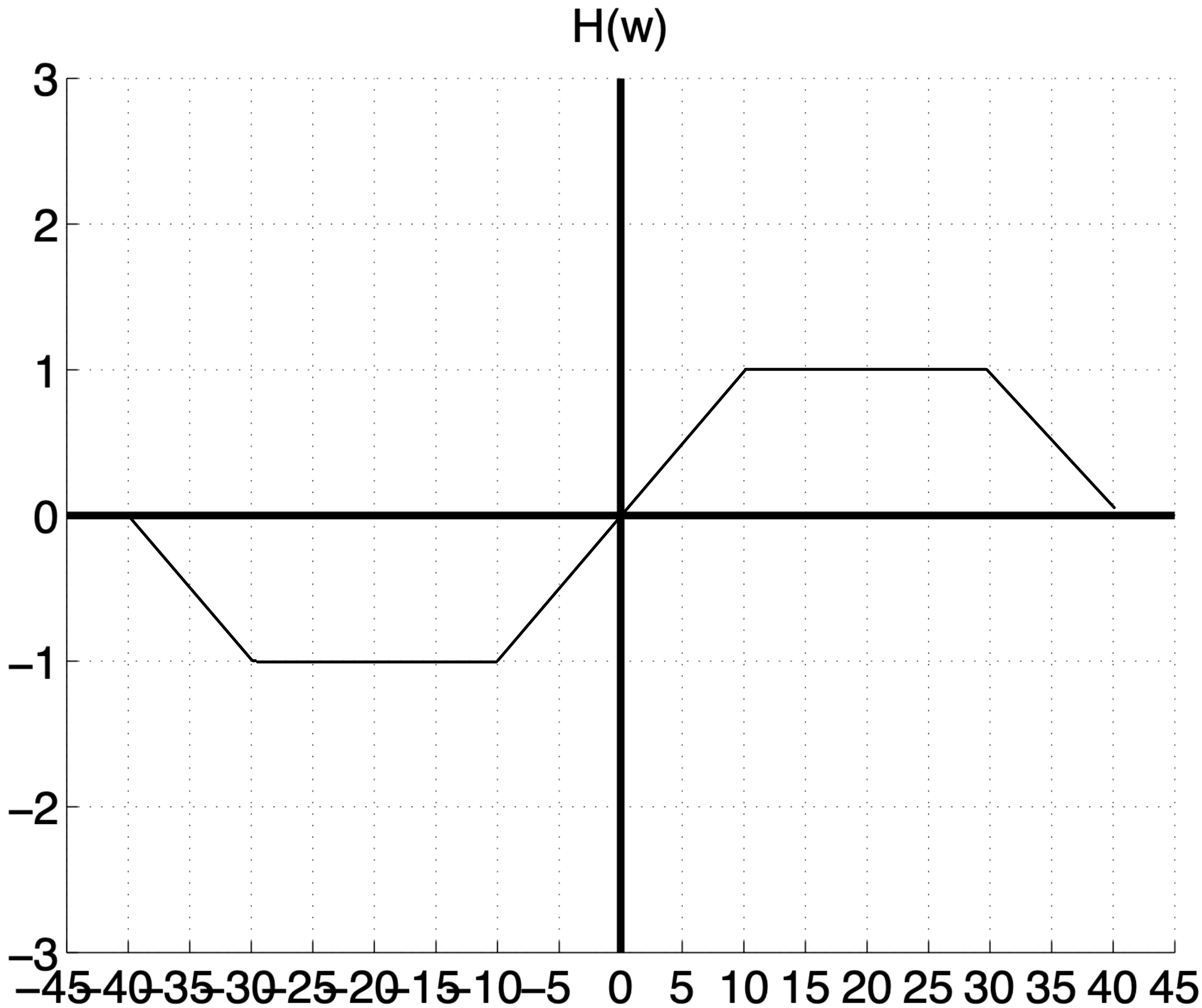


Figure 2: Frequency response of LTI system

(b) (20 points) Consider an LTI system with Frequency response shown below:

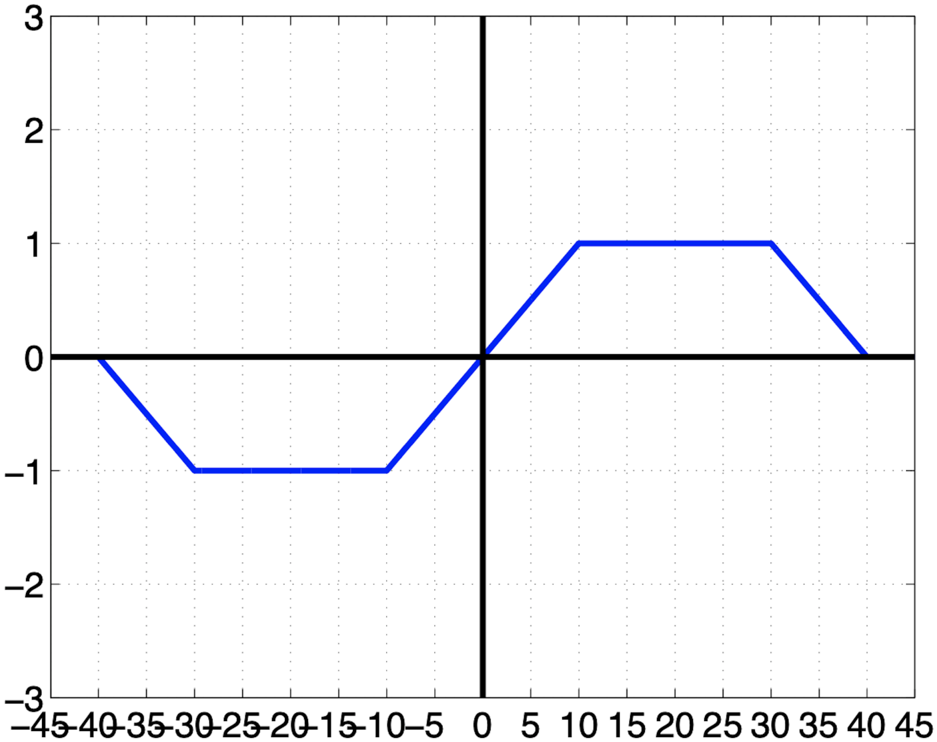


Figure 3: Frequency Response of LTI system

For the LTI system with the frequency response sketched above, determine the output $y(t)$ for the input $x(t)$ given below, which is the Fourier Series expansion for a periodic sawtooth waveform with fundamental frequency $\omega_0 = 5$ rads/sec.

$$x(t) = \sum_{k=-\infty}^{k=\infty} a_k e^{jk5t}$$

where,

$$a_k = \frac{j(-1)^k}{k\pi} \quad for k \neq 0,\ a_0 = 0$$

Show your work and write your expression for $y(t)$ in the space provided below and on the next page.

$$e^{jk\omega_0 t} \rightarrow \boxed{LTI} \rightarrow H(jk\omega_0)\, e^{jk\omega_0 t} \qquad \omega_0 = 5$$

Complex exponentials are eigenfunctions for LTI systems, and the coefficient of output is given by $H(jk\omega_0 t)$

Since $H(j\omega)$ is 0 for $|\omega| \geq 40$, we only consider the non-zero part

i.e. $|k\omega_0| < 40$

$$|k| < 8 \Rightarrow -8 < \omega < 8$$

$$a_k = \frac{j(-1)^k}{k\pi},\ k \neq 0$$

$$a_0 = 0$$

for each $a_k e^{jk5t}$ term, output is $H(j5k)\, a_k e^{jk5t}$

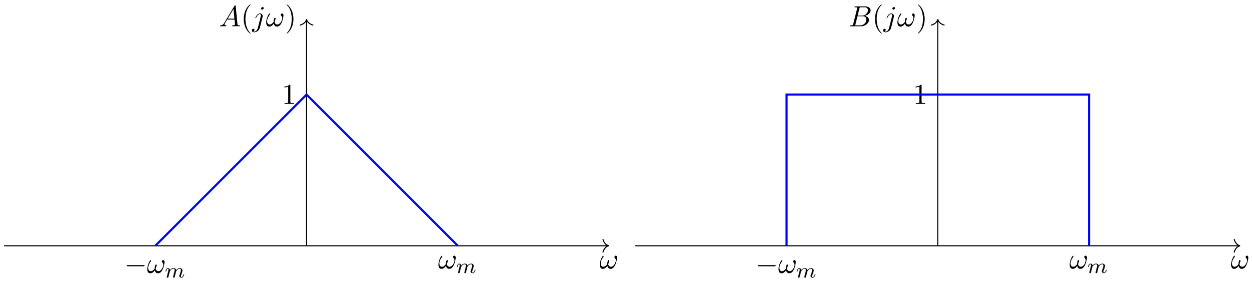
$$\Rightarrow \quad y(t) = -0.5 \cdot a_{-7} e^{-j35t} + \sum_{k=-6}^{k=-2} -a_k e^{jk5t} - 0.5 \cdot a_{-1} e^{-j5t}$$

$$+ 0.5 \cdot a_1 e^{j5t} + 1 \cdot \sum_{k=2}^{k=6} a_k e^{jk5t} + 0.5 \cdot a_7 e^{j35t}$$

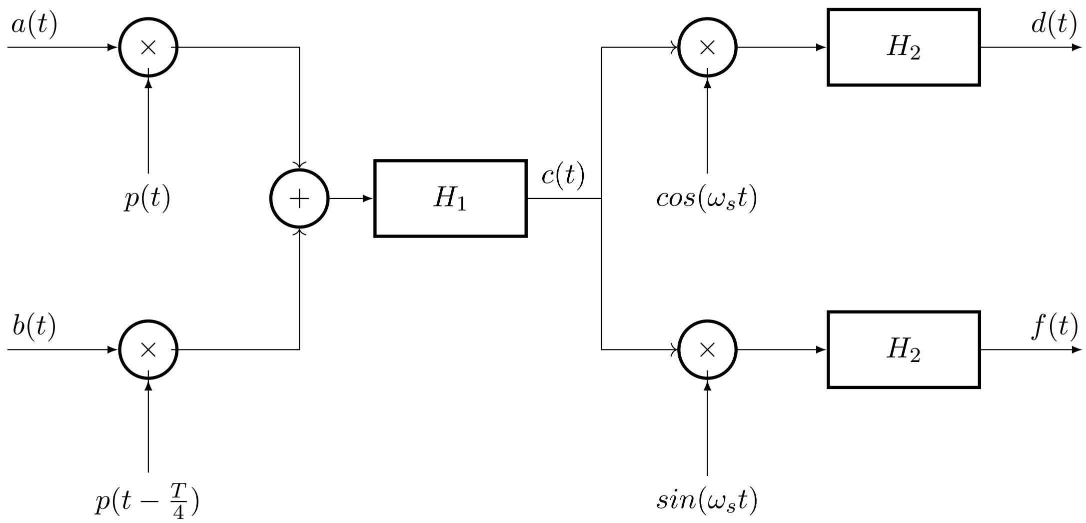
$$= -0.5 \cdot \frac{j}{7\pi} e^{-j35t} - \sum_{k=-6}^{k=-2} a_k e^{jk5t} - 0.5 \frac{j}{\pi} e^{-j5t} - 0.5 \cdot \frac{j}{\pi} e^{j5t} + \sum_{k=2}^{k=6} a_k e^{jk5t}$$

$$- 0.5 \cdot \frac{j}{7\pi} e^{j35t}$$

3. **Modulation and demodulation** (56 points)
It is given that input signals $a(t)$ and $b(t)$ are real and even, with Fourier Transforms shown below.



Consider the system below:



Where we define the following filters with conditions:

$$H_1(j\omega) = \begin{cases} T, & \omega_s - \omega_m \leq |\omega| \leq \omega_s + \omega_m \\ 0, & \text{otherwise.} \end{cases}$$

$$H_2(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_m \\ 0, & \text{otherwise.} \end{cases}$$

$$p(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$\frac{2\pi}{T} = \omega_s > 2\omega_m$$

Note: You are not required to consider amplitude scaling when answering this question.

$\omega_0 = \frac{2\pi}{T}$

(a) (15 points) In lectures, we found the Fourier Transform of the impulse train using the Fourier Series. Using a similar argument, derive the Fourier Transform of the shifted impulse train, $p(t - \frac{T}{4})$.

Hint: Recall $e^{-j\frac{\pi}{2}} = -j$. You may need this to simplify your solution.

$$p(t-\tfrac{T}{4}) = \sum_{-\infty}^{\infty} \delta(t-\tfrac{T}{4}-nT)$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t-\tfrac{T}{4})e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t-\tfrac{T}{4})e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t-\tfrac{T}{4})e^{-jk\frac{2\pi}{T}\frac{T}{4}}\,dt$$

$$= \frac{e^{-jk\frac{\pi}{2}}}{T}$$

$$p(t-\tfrac{T}{4}) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$\mathcal{F}[p(t-\tfrac{T}{4})] = \mathcal{F}\Big[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\Big]$$

$$= \sum_{k=-\infty}^{\infty} C_k \mathcal{F}[e^{jk\omega_0 t}]$$

$$= \sum_{k=-\infty}^{\infty} C_k \cdot 2\pi\delta(\omega-k\omega_0)$$

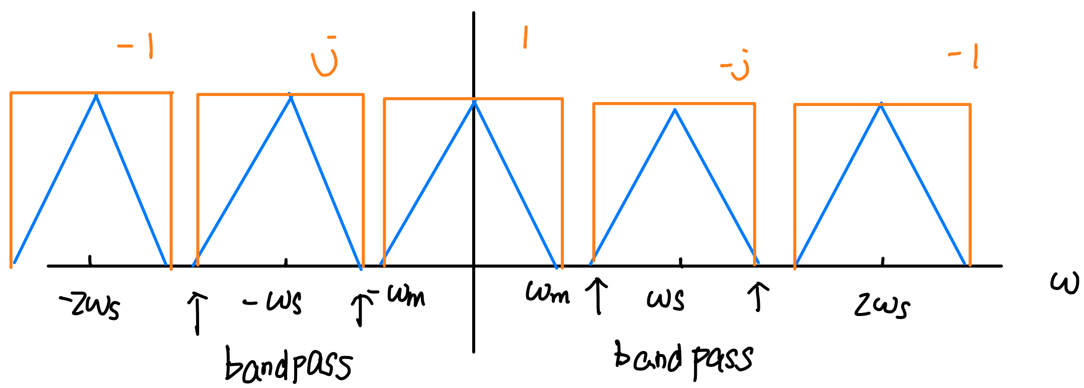
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{-jk\frac{\pi}{2}}\delta(\omega-k\omega_0)$$

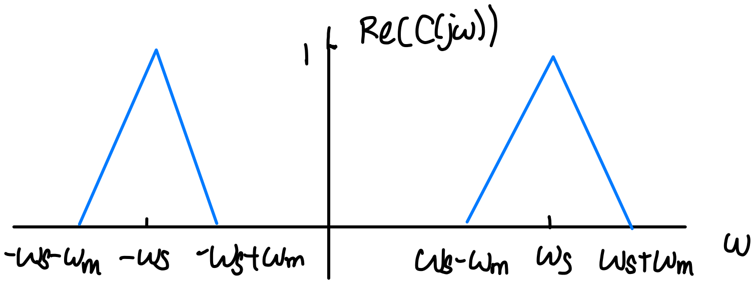
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} (-j)^k\delta(\omega-k\omega_0)$$

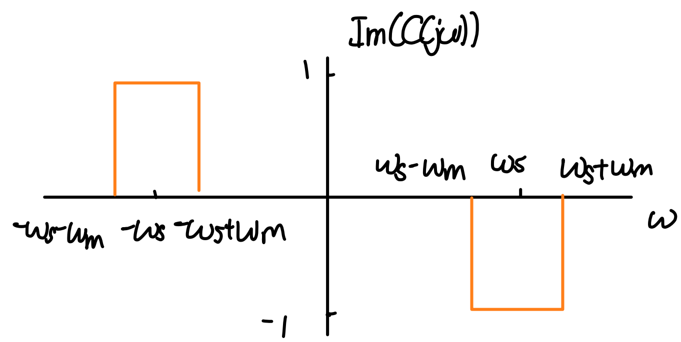
here $\omega_0 = \frac{2\pi}{T} = \omega_s$

$$\mathcal{F}[p(t-\tfrac{T}{4})] = \frac{1}{T}\sum_{k=-\infty}^{\infty} (-j)^k\delta(\omega-k\omega_s)$$

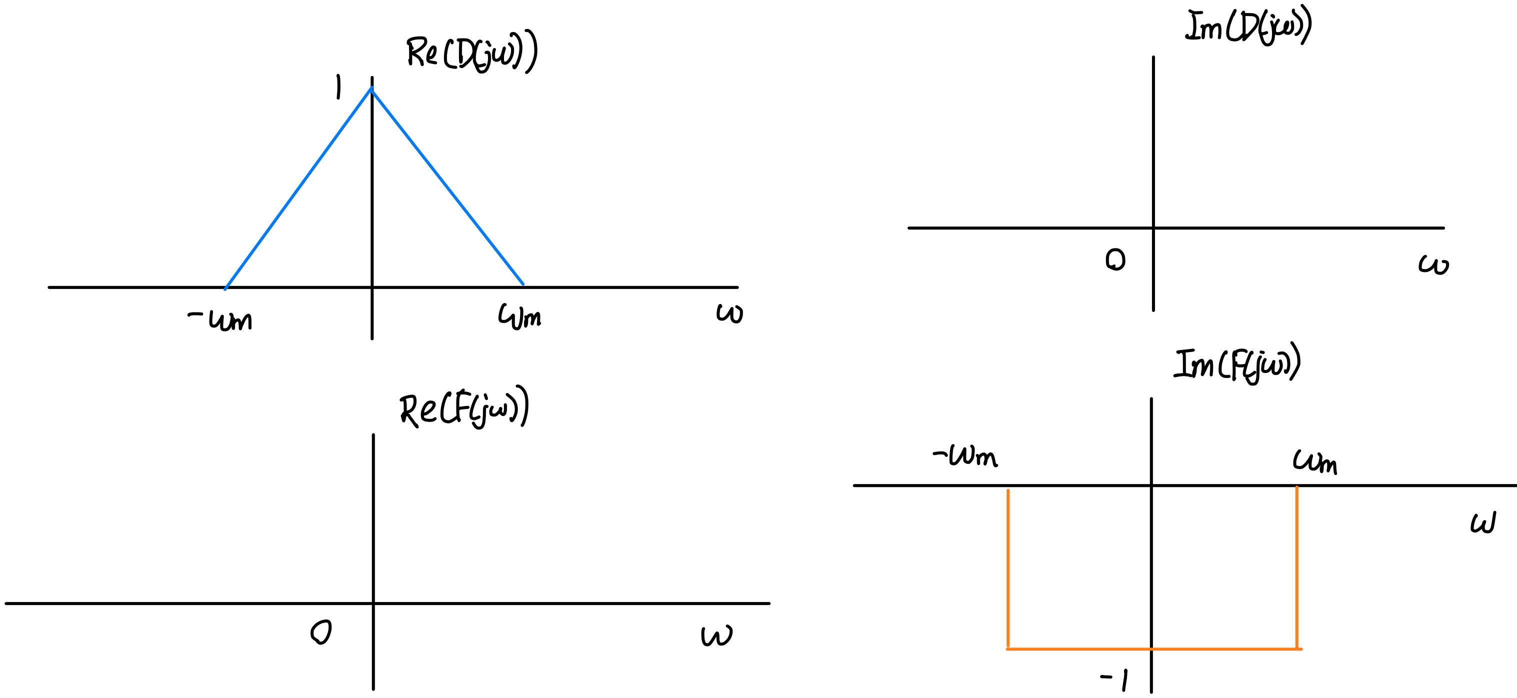
(b) (16 points) Plot the Real and Imaginary components of $C(j\omega)$ (the Fourier Transform of the output signal of the bandpass filter H_1).







(c) (16 points) Plot the Real and Imaginary Components of $D(j\omega)$ and $F(j\omega)$.



(d) (9 points) Describe how $d(t)$ relates to $a(t)$ and/or $b(t)$.

$D(j\omega)$ is the same $A(j\omega)$. $F(j\omega)$ is the same as $-B(j\omega)$

so by linearity of inverse Fourier Transform,

$d(t) = a(t)$

$f(t) = -b(t)$

4. **Laplace Transform** (46 points)

(a) (20 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. $f(t) = e^{-at}\underbrace{(\cos(\omega_0 t) - 1)\,u(t)}$

$= \cos(\omega_0 t)u(t) - u(t)$

$\mathcal{L}[\cos(\omega_0 t)u(t) - u(t)] = \frac{s}{s^2+\omega_0^2} - \frac{1}{s}$ $\quad Re(s) > 0$

also, $\mathcal{L}[e^{at}g(t)] = G(s-a)$ by s-shift property

$\Rightarrow F(s) = \mathcal{L}[e^{-at}(\cos(\omega_0 t) - 1)u(t)]$

$= \frac{s+a}{(s+a)^2+\omega_0^2} - \frac{1}{s+a}$ $\quad Re(s+a) > 0$

$Re(s) > -Re(a)$

ii. $f(t) = \int_{0^-}^{t-1} \underbrace{e^{-ax}(\cos(\omega_0 x) - 1)}_{g(x)}\, dx,\ \ t - 1 \geq 0$

from (i) we already know $\mathcal{L}[g(x)] = \frac{s+a}{(s+a)^2+\omega_0^2} - \frac{1}{s+a} = G(s)$

$f(t+1) = \int_0^t g(x)dx$

$\mathcal{L}[f(t+1)] = \frac{G(s)}{s} = e^s F(s)$

$F(s) = \frac{G(s)}{se^s} = \frac{e^{-s}}{s}\left(\frac{s+a}{(s+a)^2+\omega_0^2} - \frac{1}{s+a}\right)$ $\quad Re(s) > 0$

(b) (10 points) Find the inverse Laplace transform $f(t)$ for the following $F(s)$. ($f(t)$ is a causal signal.)

$$F(s) = \frac{s^2+s+1}{(s+1)(s+2)(s+3)}$$

let $F(s) = \frac{r_1}{s+1} + \frac{r_2}{s+2} + \frac{r_3}{s+3}$

$r_1 = F(s)(s+1)\Big|_{s=-1} = \frac{1-1+1}{1\cdot 2} = \frac{1}{2}$

$r_2 = F(s)(s+2)\Big|_{s=2} = \frac{4-2+1}{-1\cdot 1} = -3$

$r_3 = F(s)(s+3)\Big|_{s=-3} = \frac{9-3+1}{(-2)\cdot(-1)} = \frac{7}{2}$

$F(s) = \frac{1/2}{s+1} - \frac{3}{s+2} + \frac{7/2}{s+3}$

$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}e^{-t} - 3e^{-2t} + \frac{7}{2}e^{-3t}$

(c) (16 points) Prove the following statements

i. (8 points) If $x(t)$ is an even function, so that $x(t) = x(-t)$, then $X(s) = X(-s)$.

$$X(-s) = \int_{-\infty}^{\infty} x(t) e^{st} dt$$

let $t' = -t$
$dt' = -dt$

$$= \int_{\infty}^{-\infty} x(-t') e^{-st'} (-dt')$$

$x(t') = x(-t')$ since x is even

$$= \int_{-\infty}^{\infty} x(t') e^{-st'} dt' = X(s)$$

$$\Rightarrow X(-s) = X(s)$$

ii. (8 points) If $x(t)$ is an odd function, so that $x(t) = -x(-t)$, then $X(s) = -X(-s)$.

$$X(-s) = \int_{-\infty}^{\infty} x(t) e^{st} dt$$

let $t' = -t$
$dt' = -dt$

$$= \int_{\infty}^{-\infty} x(-t') e^{-st'} (-dt')$$

$x(t') = -x(-t')$ since x is odd

$$= \int_{-\infty}^{\infty} -x(t') e^{-st'} dt'$$

$$= -\int_{-\infty}^{\infty} x(t') e^{-st'} dt' = -X(s)$$

$$\Rightarrow X(s) = -X(-s)$$

Bonus (15 points)

This problem is about determining the Fourier Transform of the signal $x_a(t) = \tan^{-1}(at)$, working through a sequence of successive steps. Clearly delineate your work and circle your answer for each part below.

1. (10 points) Compute the Fourier transform of

$$x(t) = \tan^{-1}(t)$$

Hint: You might find the following calculus result useful

$$\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$$

2. (3 points) Does your answer for part (a) have a real part? Explain how your answer is consistent with the symmetry properties of the Fourier Transform.

3. (2 points) Now use one of the properties of the Fourier Transform to determine the Fourier Transform for the more general case below, where a is a real-valued positive constant.

$$x_a(t) = \tan^{-1}(at)$$