ECE102, Fall 2019
Department of Electrical and Computer Engineering
University of California, Los Angeles

Final
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UCLA True Bruin academic integrity principles apply.
Open: Four cheat sheets allowed.
Closed: Book, computer, internet.
8:00-11:00am.
Wednesday, 11 Dec 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: Shaodian Wang

Signature: [signature]

ID#: 305111459

Problem 1	____	/ 40
Problem 2	____	/ 45
Problem 3	____	/ 40
Problem 4	____	/ 30
Problem 5	____	/ 45
BONUS	____	/ 15 bonus points
Total	____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (40 points)

   (a) (16 points) For each statement below, determine whether it is true or false. You must justify your answer to receive full credit.

   i. (8 points) If $f(t)$ is a real and even signal, and $g(t)$ is a real and odd signal, the convolution of $f(t)$ and $g(t)$ is real and odd.

True.

$f(t)$ real & even $\Rightarrow F(j\omega)$ real & even.

$g(t)$ real & odd $\Rightarrow G(j\omega)$ odd & imaginary

Let $x(t) = f(t) * g(t)$.

$X(j\omega) = F(j\omega) \cdot G(j\omega)$

which will be imaginary & odd

$\Rightarrow x(t)$ will be real & odd

   ii. (8 points) All LTI systems are stable.

False. Consider the LTI system with impulse response $h(t) = u(t)$.

The output $y(t) = x(t) * u(t)$ will not be bounded even if the input signal $x(t)$ is bounded

(b) (12 Points) Suppose we have an unknown system (black box). We input

$$x(t) = \text{sinc}(t)$$

into the system, and measure that its output is

$$y(t) = e^{-t}u(t).$$

Can this system be LTI? You must justify your answer to receive full credit.

No

Time invariant:

If we input $x(t-T)$, i.e. $\text{sinc}(t-T)$,

we should expect the output to be

$y(t-T) = e^{-(t-T)}u(t-T)$

Assume $H(jw)$ is the frequency response of the system

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{1}{(1+jw)\,\text{rect}\left(\frac{w}{2\pi}\right)}$$ which is

infinite for $|w| > \pi$

$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)e^{jwt}\,dw$ is hence

infinite for all t

Since the impulse response is infinite, but impulse responses should exist for all LTI systems, this cannot be LTI

You can write sinc(t) as a continuous sum of complex exponentials from $|w| < \pi$.

The fourier transform $Y(jw)$ of $y(t)$, however, is $\frac{1}{1+jw}$, and has non zero values beyond this range. Since complex exponentials are eigenfunctions of LTI systems, this does not make sense, and this system introduces frequencies beyond that of the input. Hence, it is not LTI.

(c) (12 Points) Determine whether the following system is (1) causal, and (2) stable.

$$y(t) = \int_{-\infty}^{t} (x(\tau) + e^{-\tau})u(\tau + 1)d\tau$$

The system is causal.

$y(t)$ does not depend on future inputs of $x(t)$,

it only depends on past and present inputs, where $\tau \leq t$

The system is not stable.

Assuming $|x(\tau) + e^{-\tau}| < M_x$ for $-1 \leq \tau < \infty$

$$y(t) < \int_{-\infty}^{t} u(\tau+1) M_x \, d\tau$$

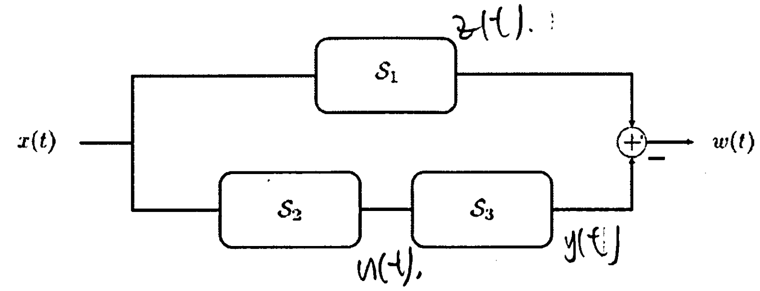
$$= M_x \int_{-1}^{t} 1 \, d\tau$$

$$= M_x (t+1).$$

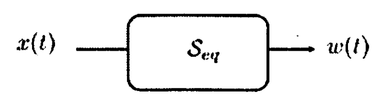
But as $t \to \infty$, $y(t) < \infty$,

meaning that it is not bounded

2. **Frequency Response and LTI system** (45 points)
Suppose the three systems are interconnected as shown below.



And we denote the equivalent system as below.



(a) (8 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all LTI systems. Is the equivalent system $\mathcal{S}_{eq}$ an LTI system? Please justify your answer to receive full credit.

Linearity:

Let $x_3(t) = a x_1(t) + b x_2(t)$.

Since S_1 is LTI, $z_3(t) = a z_1(t) + b z_2(t)$.

Since S_2 is LTI, $u_3(t) = a u_1(t) + b u_2(t)$.

Since S_3 is LTI, $y_3(t) = a y_1(t) + b y_2(t)$

$w_3(t) = z_3(t) - y_3(t) = a(-y_1(t) + z_1(t)) + b(z_2(t) - y_2(t))$

$= a w_1(t) + b(w_2(t)) \Rightarrow$ system is linear.

Time-invariance: $x(t-T) \xrightarrow{S_2} u(t-T) \xrightarrow{S_3} y(t-T)$ since S_2, S_3 are LTI.

$x(t-T) \xrightarrow{S_1} z(t-T)$ since S_1 is LTI.

$\Rightarrow w(t-T) = S_3[S_2[x(t-T)]] - S_1 x(t-T) \Rightarrow$ system is time invariant.

$\Rightarrow S_{eq}$ is LTI.

(b) (8 points) Suppose the equivalent system $\mathcal{S}_{eq}$ is an LTI system. Are $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ all necessarily LTI systems? Please justify your answer to receive full credit.

No. S_2 & S_3 need not be LTI.

Consider $u(t) = x(2t)$ for S_2 & $y(t) = x(\frac{1}{2}t)$ for S_3.

Both S_2 & S_3 are hence not time-invariant.

However, passing $x(t)$ through S_2 and then S_3 would recover $x(t)$ and hence make S_{eq} an LTI system if S_1 is LTI.

(c) (15 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are each characterized by an LTI system,

- The first system $\mathcal{S}_1$, with frequency response $H_1(j\omega)$, is given by its input-output relationship: $y(t) = x(t-3)$;
- The second system $\mathcal{S}_2$, with frequency response $H_2(j\omega)$, is given by its impulse response: $h_2(t) = u(t-3)$;
- The third system $\mathcal{S}_3$, with frequency response $H_3(j\omega)$, is given by its input-output relationship: $y(t) = \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$.

Determine the frequency responses $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ of each system as well as $H_{eq}(j\omega)$ of the equivalent system.

$H_1(j\omega)$: $\quad H_1(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{X(j\omega)e^{-3j\omega}}{X(j\omega)} = e^{-3j\omega}$

$H_2(j\omega)$: $\quad h_2(t) = u(t-3) \Longleftrightarrow \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)e^{-3j\omega}$

$H_3(j\omega)$: $\quad Y(j\omega) = j\omega X(j\omega) + (j\omega)^2 X(j\omega)$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = j\omega(1+j\omega).$$

$$H_{eq} = H_1 - H_2H_3$$

$$= e^{-3j\omega} - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)e^{-3j\omega}\left(j\omega + (j\omega)^2\right)$$

$$= e^{-3j\omega}\left(1 - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\left(j\omega + (j\omega)^2\right)\right)$$

$$= e^{-3j\omega}\left(1 - \left(\pi\delta(\omega)\left(j(0) + j(0)^2\right) + 1 + j\omega\right)\right)$$

$$= e^{-3j\omega}\left(j\omega\right)$$

(d) (14 points) For the system in part(c), the output $w(t)$ to an input $x(t) = e^{j\pi t/3}$ can be written as:

$$w(t) = Ae^{j\theta}x(t).$$

Determine A and θ.

$$e^{j\pi t/3} \Longleftrightarrow 2\pi\,\delta(\omega - \pi/3).$$

$$W(j\omega) = 2\pi\,\delta(\omega - \tfrac{\pi}{3})\,H_{e_\delta}(j\omega)$$

$$= 2\pi\,\delta(\omega - \tfrac{\pi}{3})\,e^{-3j\omega}(j\omega).$$

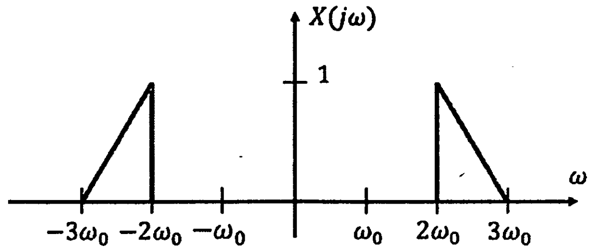
$$= 2\pi\,\delta(\omega - \tfrac{\pi}{3})\left(e^{-\pi j}\right)\left(\tfrac{\pi}{3}j\right)$$

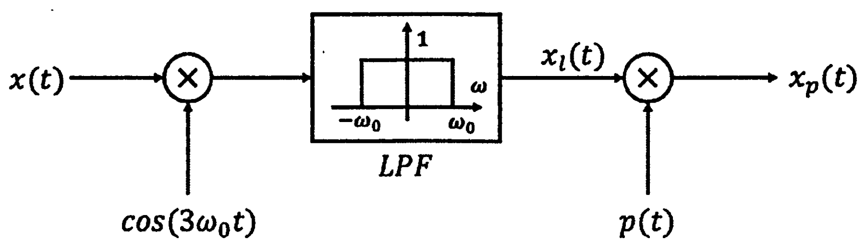
$$= \frac{2\pi^2}{3}(-1)\left(e^{j\frac{\pi}{2}}\right)\delta(\omega - \tfrac{\pi}{3}).$$

$$F^{-1}(W(j\omega)) = -\frac{\pi}{3}\,e^{j\frac{\pi}{2}}\,e^{jt\pi/3}$$

$$\Rightarrow A = -\frac{\pi}{3},\quad \theta = \pi/2.$$

3. **Sampling and Modulation** (40 points)
Assume we have a continuous bandpass signal $x(t)$ with frequency spectrum as shown below. We also assume that $x(t)$ is real. The sampling theorem states that, to recover a signal without distortion, a signal must be sampled at a rate greater than twice its bandwidth. However, since $x(t)$ has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than the Nyquist rate could be used. Now consider the system shown below where $p(t)$ is the sampling function.





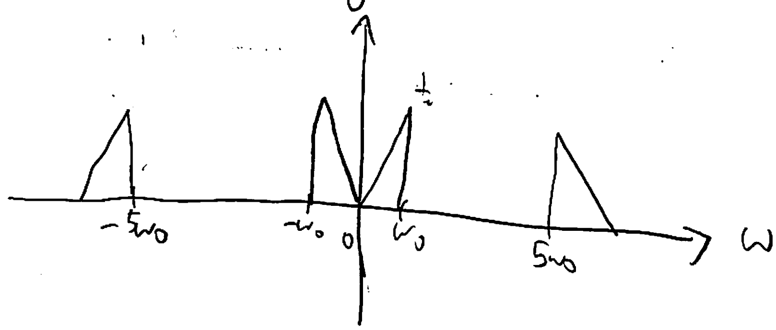
(a) (5 points) What is the Nyquist rate of $x(t)$?

Nyquist rate : $\frac{6\omega_0}{2\pi} = \frac{3\omega_0}{\pi}$

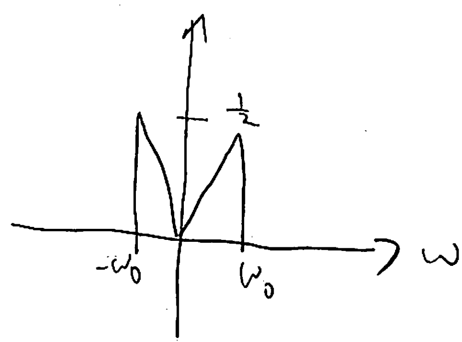
(b) (5 points) What is the Nyquist rate of $x_l(t)$? Sketch the frequency spectrum after the low pass filter, i.e. $X_l(j\omega)$.

$$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}\left[X(j(\omega-3\omega_0)) + X(j(\omega+3\omega_0))\right].$$

With frequency spectrum



After the LPF,



$\frac{2\pi}{\frac{\pi}{2\omega_0}}$ $4\omega_0$

(c) (10 points) If the sampling function is an impulse train

$T\uparrow$ $\omega_0\downarrow$

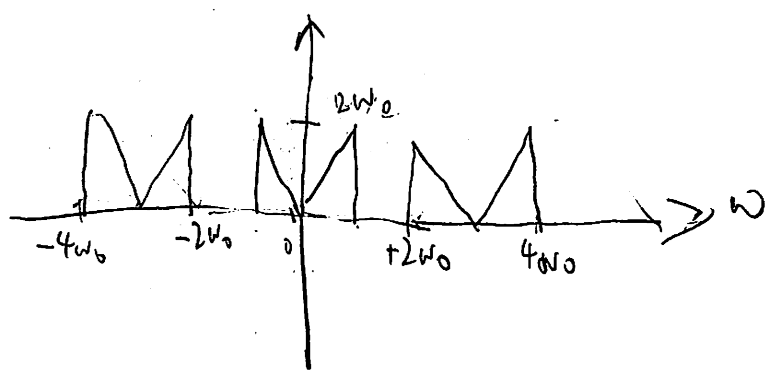
$$p(t) = \sum_{k=-\infty}^{k=+\infty} \delta(t - kT)$$

find the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$. Sketch the output frequency spectrum $X_p(jw)$.

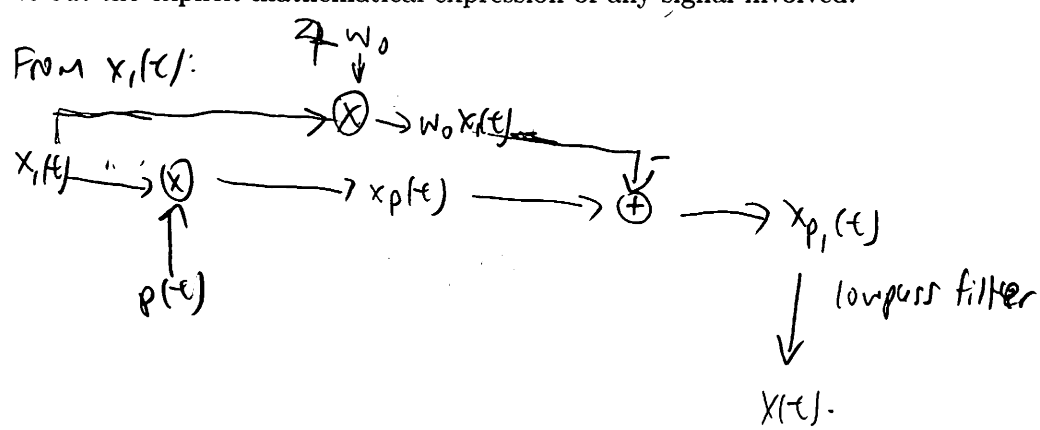
$$p(t) = \delta_T(t) \Longleftrightarrow \frac{2\pi}{T}\delta_{\frac{2\pi}{T}}(\omega)$$

The max sampling period is ~~when~~ sampling rate is the Nyquist rate of $x_1(t) = \frac{2\omega_0}{\pi}$ and T is here $T = \frac{1}{f} = \frac{\pi}{2\omega_0}$

Output:



(d) (20 points) With the $p(t)$ found in part (c), design a system to recover $x(t)$ from $x_p(t)$ without using a bandpass or highpass filter. Note that the recovered signal should have the same amplitude as $x(t)$ in frequency spectrum. Draw a flow diagram of your system and clearly state each component (including cutoff frequencies of any lowpass filter). Write out the explicit mathematical expression of any signal involved.



The low pass filter will be defined as $\frac{1}{2w_0} \mathrm{rect}\left(\frac{w}{6w_0}\right)$

4. **Laplace Transform** (30 points)

A system can be described by the following differential equation:

$$y''(t) + y'(t) - 2y(t) = 6x'(t) - 3x(t)$$

where the initial conditions are all zero, i.e. $y''(0) = 0$, $y'(0) = 0$ and $y(0) = 0$.

(a) (10 points) Find the transfer function $H(s) = Y(s)/X(s)$. Assume $x(0) = 0$.

$$s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) - 2Y(s) = 6(s X(s) - x(0)) - 3X(s).$$

$$s^2 Y(s) + sY(s) - 2Y(s) = 6s(X(s)) - 3X(s).$$

$$Y(s)(s^2+s-2) = X(s)(6s-3).$$

$$\frac{Y(s)}{X(s)} = \frac{6s-3}{s^2+s-2} \Rightarrow \frac{6s-3}{(s+2)(s-1)} = \frac{r_1}{s+2} + \frac{r_2}{s-1}.$$

To find r_1, $\left.\frac{(6s-3)}{(s-1)}\right|_{s=-2} = r_1 = \frac{-12-3}{-3} = 5$

r_2: $\left.\frac{6s-3}{s+2}\right|_{s=1} = r_2 = 1.$

$$\Rightarrow H(s) = \frac{5}{s+2} + \frac{1}{s-1}.$$

(b) (20 points) If the input is

$$x(t) = e^{-t}u(t)$$

then find the output $y(t)$.

$$X(s) = \frac{1}{s+1}$$

$$\Rightarrow Y(s) = X(s)\,H(s)$$

$$= \left(\frac{1}{s+1}\right)\left(\frac{5}{s+2} + \frac{1}{s-1}\right)$$

$$= \frac{5}{(s+2)(s+1)} + \frac{1}{(s+1)(s-1)}$$

For $\frac{5}{(s+2)(s+1)} = \frac{r_1}{s+2} + \frac{r_2}{s+1}$

$r_1 : \frac{5}{s+1}\Big|_{s=-2} = -5 \qquad r_2 : \frac{5}{s+2}\Big|_{s=-1} = 5$

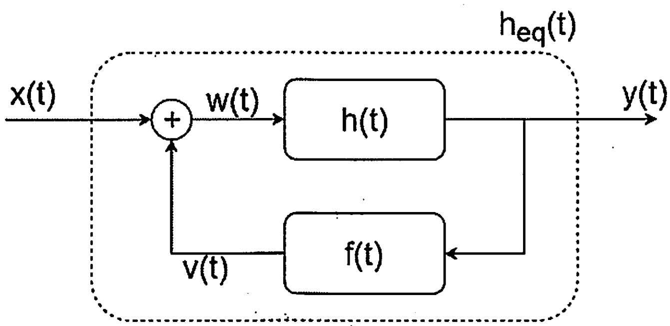
For $\frac{1}{(s+1)(s-1)} = \frac{r_1}{s+1} + \frac{r_2}{s-1}$

$r_1 = \frac{1}{s-1}\Big|_{s=-1} = \frac{1}{-2} \qquad r_2 = \frac{1}{s+1}\Big|_{s=1} = \frac{1}{2}$

$$\Rightarrow Y(s) = -\frac{5}{s+2} + \frac{5}{s+1} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

$$\mathcal{L}^{-1}(Y(s)) = -5e^{-2t}u(t) + 5e^{-t}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(t)$$

5. **Feedback System** (45 points)
Consider the feedback system shown below (all components are LTI):



where $h(t) = e^{-2t}u(t)$ and $y(0) = 0$.

(a) (10 points) Show that

$$H_{eq}(s) = \frac{H(s)}{1 - H(s)F(s)}$$

$$Y(s) = (X(s) + F(s)Y(s))\,H(s).$$

$$Y(s) - F(s)H(s)Y(s) = X(s)H(s).$$

$$Y(s) = \frac{X(s)H(s)}{1 - H(s)F(s)}$$

$$\Rightarrow H_{eq}(s) = \frac{H(s)}{1-H(s)F(s)}$$

(b) (10 points) Find the Laplace Transform $H(s)$ of $h(t)$. What is the frequency response $H(j\omega)$? Why is this a low-pass filter?

$$h(t) \Leftrightarrow \frac{1}{s+2} = H(s).$$

$$H(j\omega) = H(s)\Big|_{s=j\omega} = \frac{1}{j\omega+2}$$, valid because $Re\{s\} > -2$

$$|H(j\omega)| = \left| \frac{-j\omega+2}{2^2+\omega^2} \right| = \sqrt{\frac{2^2+\omega^2}{(2^2+\omega^2)^2}} = \sqrt{\frac{1}{4+\omega^2}}$$

This is a low pass filter because at $|\omega| >> 4$,

$|H(j\omega)| \approx 0$ and at $|\omega| \to 0$, $|H(j\omega)| \approx \frac{1}{2}$

(c) (10 points) $v(t)$ and $y(t)$ satisfy the differential equation

$$v(t) = \frac{d}{dt}y(t) + y(t) - 10\int_0^t y(\tau)d\tau$$

What is $F(s)$?

$$V(s) = sY(s) - y(0) + Y(s) - 10\frac{Y(s)}{s}$$

$$V(s) = Y(s)\left(s+1-\frac{10}{s}\right)$$

$$F(s) = \frac{V(s)}{Y(s)} = s+1-\frac{10}{s}$$

(d) (15 points) Using $F(s)$ found in part c, what is $h_{eq}(t)$? Is this a low-pass, band-pass, or high-pass filter?

$$H_{eq}(s) = \frac{H(s)}{1-H(s)F(s)} = \frac{\frac{1}{s+2}}{\frac{s+2-(s+1-\frac{10}{s})}{s+2}}$$

$$= \frac{1}{1+\frac{10}{s}}$$

$$= \frac{s}{s+10} = 1 - \frac{10}{s+10}.$$

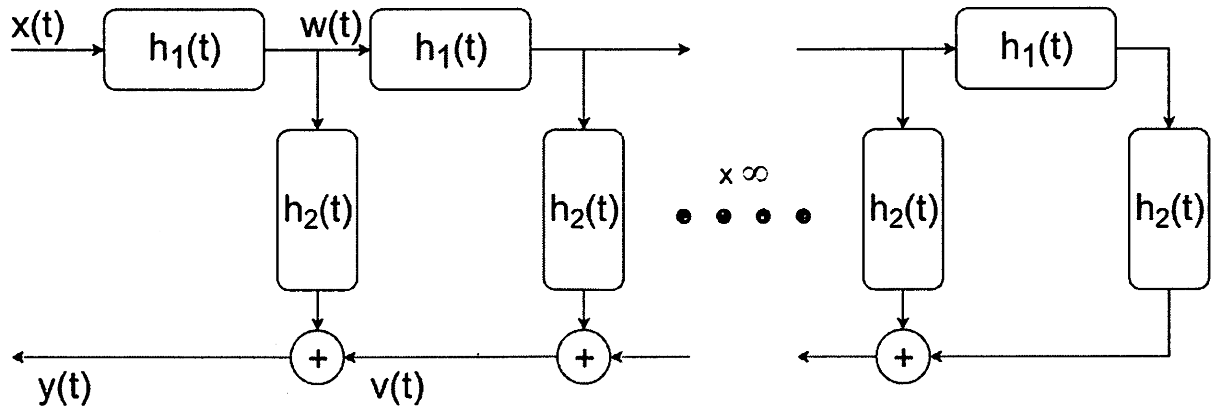
$$h_{eq}(t) = \mathcal{L}^{-1}\left(1 - \frac{10}{s+10}\right) = \delta(t) - 10\, e^{-10t} u(t).$$

This is a high-pass filter, because

at $|\omega| \to 0$, $H_{eq}(s) \approx 0$,

and at $|\omega| \gg 10$, $H_{eq}(s) \approx 1$

Bonus (15 points) Consider the LTI system S shown below, which is a system ladder with an infinite number of rungs. Let $y(t) = S[x(t)]$.



(a) (8 points) In terms of $H_1(s)$ and $H_2(s)$, what is the equivalent transfer function $H_{eq}(s)$ between $Y(s)$ and $X(s)$? *Hint: how does* $\frac{V(s)}{W(s)}$ *relate to* $\frac{Y(s)}{X(s)}$*?*

$$W(s)H_2(s) + V(s) = Y(s).$$

$$W(s) = X(s)H_1(s).$$

$$\frac{V(s)}{W(s)} = H_{eq}(s).$$

$$\Rightarrow \quad W(s)H_2(s) + W(s)H_{eq}(s) = Y(s).$$

$$\Rightarrow \; Y(s)\,H_1(s)\,(H_2(s) + H_{eq}(s)) = Y(s).$$

$$\Rightarrow \; H_{eq}(s) = H_1(s)\;(H_2(s) + H_{eq}(s))$$

$$H_{eq}(s) - H_1(s)H_{eq}(s) = H_1(s)H_2(s).$$

$$H_{eq}(s) = \frac{H_1(s)H_2(s)}{1 - H_1(s)}.$$

(b) (7 points) Suppose $h_1(t) = e^{-a_1 t}u(t)$ and $h_2(t) = e^{-a_2 t}u(t)$, where a_1 and a_2 are real and positive. For what values of a_1 is S BIBO stable?

$$H_1(s) = \frac{1}{s+a_1}, \quad H_2(s) = \frac{1}{s+a_2}$$

For S to ~~be~~ BIBO stable,

$$H_{eq}(s) = \frac{\left(\frac{1}{s+a_1}\right)\left(\frac{1}{s+a_2}\right)}{1 - \frac{1}{s+a_1}}$$

$$= \frac{\frac{1}{(s+a_1)(s+a_2)}}{\frac{s+a_1-1}{s+a_1}}$$

$$= \frac{1}{(s+a_2)(s+a_1-1)}$$