ECE102, Fall 2019
Department of Electrical and Computer Engineering
University of California, Los Angeles

Final
Prof. J.C.Kao
TAs: W. Feng, J. Lee & S. Wu

UCLA True Bruin academic integrity principles apply.
Open: Four cheat sheets allowed.
Closed: Book, computer, internet.
8:00-11:00am.
Wednesday, 11 Dec 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: Justin Jianto

Signature: ______________

ID#: 905069023

Problem 1	____	/ 40
Problem 2	____	/ 45
Problem 3	____	/ 40
Problem 4	____	/ 30
Problem 5	____	/ 45
BONUS	____	/ 15 bonus points
Total	____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (40 points)

   (a) (16 points) For each statement below, determine whether it is true or false. You must justify your answer to receive full credit.

      i. (8 points) If $f(t)$ is a real and even signal, and $g(t)$ is a real and odd signal, the convolution of $f(t)$ and $g(t)$ is real and odd.

still even $f(t) * g(t)$ flip & drag

$\int f(t-\tau)\, g(\tau)\, d\tau$ $\boxed{\text{True}}$

$F(j\omega)G(j\omega)$] odd

even odd

      ii. (8 points) All LTI systems are stable.

$S(ax + b\tilde{x}) = a\,S(x) + b\,S(\tilde{x})$ $\int_{-\infty}^{t} |x(t)|\,dt$

True, since we're able to convolute it, it must converge

x^2

(b) (12 Points) Suppose we have an unknown system (black box). We input

$$x(t) = \text{sinc}(t)$$

into the system, and measure that its output is

$$y(t) = e^{-t}u(t).$$

Can this system be LTI? You must justify your answer to receive full credit.

$$x(t) \to \boxed{LTI} \to e^{-t}u(t)$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) \qquad Y(j\omega) = \frac{1}{1+j\omega}$$

$$\frac{\frac{1}{1+j\omega}}{2\pi\, \text{rect}\left(\frac{\omega}{2\pi}\right)} = H(j\omega) \qquad \text{sinc}(t) = 2\pi\, \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$H(j\omega) = \frac{1}{(1+j\omega)\, 2\pi\, \text{rect}\left(\frac{\omega}{2\pi}\right)}$$

Yes, since we can find a frequency response, there must be 1 case when it has a h(t), thus LTI.

(c) (12 Points) Determine whether the following system is (1) causal, and (2) stable.

$$y(t) = \int_{-\infty}^{t} (x(\tau) + e^{-\tau})u(\tau + 1)d\tau$$

$$y(t) = \int_{-\infty}^{t} x(\tau)u(\tau+1)d\tau + \int_{-\infty}^{t} e^{-\tau}u(\tau+1)d\tau$$

$$y(t) = \int_{-1}^{t} x(\tau) + e^{-\tau} d\tau$$

$$= \int_{-1}^{t} x(\tau)\,d\tau + \int_{-1}^{t} e^{-\tau} d\tau$$

1) not causal, cannot exist before $t=0$

2) Assume $|x(t)| \le M_x \le \infty$

$$y(t) = \left| \int_{-1}^{t} x(\tau)\,d\tau + \int_{-1}^{t} e^{-\tau} d\tau \right|$$

$$\left|\int_{-1}^{t} x(\tau)\,d\tau\right| + \left|\int_{-1}^{t} e^{-\tau} d\tau\right|$$

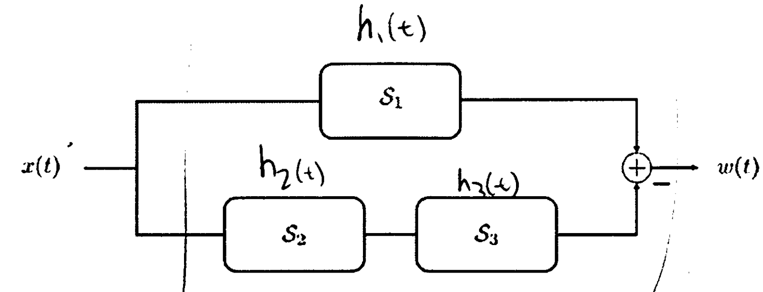
$$\int_{-1}^{t} |x(\tau)|\,d\tau + \left|\int_{-1}^{t} e^{-\tau} d\tau\right| \qquad e^{-\tau} t$$

stable

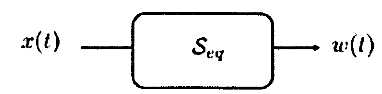
converges

$\boxed{\therefore \text{ stable}}$

2. **Frequency Response and LTI system** (45 points)
Suppose the three systems are interconnected as shown below.



And we denote the equivalent system as below.



(a) (8 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all LTI systems. Is the equivalent system $\mathcal{S}_{eq}$ an LTI system? Please justify your answer to receive full credit.

$$S_1(x(t)) + S_3(S_2(x(t))) = w(t)$$

Suppose: $ax(t)$

$$S_1(ax(t)) + S_3(S_2(ax(t))) = aS_1(x(t)) + aS_3(S_2(x(t)))$$
$$= a\left(S_1(x(t)) + S_3(S_2(x(t)))\right)$$

$\therefore$ Homogeneity

Suppose $x + \tilde{x}$

$$S_1(x+\tilde{x}) + S_3(S_2(x+\tilde{x})) = S_1(x) + S_1(\tilde{x}) + S_3(S_2(x) + S_2(\tilde{x}))$$
$$= S_1(x) + S_1(\tilde{x}) + S_3(S_2(x)) + S_3(S_2(\tilde{x}))$$

$\therefore$ Superposition

$\therefore$ Linear

(b) (8 points) Suppose the equivalent system $\mathcal{S}_{eq}$ is an LTI system. Are $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ all necessarily LTI systems? Please justify your answer to receive full credit.

No not necessarily

$$S_{eq}(ax + b\tilde{x}) = a\, S_{eq}(x) + b\, S_{eq}(\tilde{x})$$

Imagine having a sin & cos function

$S_2 = \cos^{-1}(x)$ $S_3 = \cos x$, NOT LTI

But when put together $\cos^{-1}(\cos x) = x$,

becomes LTI

(c) (15 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are each characterized by an LTI system,

- The first system $\mathcal{S}_1$, with frequency response $H_1(j\omega)$, is given by its input-output relationship: $y(t) = x(t-3)$;
- The second system $\mathcal{S}_2$, with frequency response $H_2(j\omega)$, is given by its impulse response: $h_2(t) = u(t-3)$;
- The third system $\mathcal{S}_3$, with frequency response $H_3(j\omega)$, is given by its input-output relationship: $y(t) = \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$.

Determine the frequency responses $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ of each system as well as $H_{eq}(j\omega)$ of the equivalent system.

$\underline{H_1(jw)}$

$$Y(jw) = X(jw)\,H(jw)$$
$$X(jw)\,e^{-3jw} = X(jw)\,H_1(jw)$$
$$\boxed{e^{-3jw} = H_1(jw)}$$

$H_{eq} = H_2(jw)\,H_3(jw) + H_1(jw)$

$$\boxed{H_{eq}(jw) = \left(\pi\delta(w) + \frac{1}{jw}\right)e^{-3jw}\left(jw + (jw)^2\right) + e^{-3jw}}$$

$\underline{H_2(jw)}$

$$h_2(t) = u(t-3)$$
$$H_2(jw) = \left(\pi\delta(w) + \frac{1}{jw}\right)e^{-3jw}$$

$\underline{H_3(jw)}$

$$y(t) = \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$$
$$Y(jw) = jw\,X(jw) + (jw)^2\,X(jw)$$
$$H(jw) = jw + (jw)^2$$

(d) (14 points) For the system in part(c), the output $w(t)$ to an input $x(t) = e^{j\pi t/3}$ can be written as:

$$w(t) = Ae^{j\theta}x(t).$$

Determine A and θ.

$$H_{eq}(jw) = e^{-3jw}\left(\pi\delta(w) + \frac{1}{jw}\right)\left(jw + (jw)^2\right) + 1$$

$$= e^{-3jw}\left(\pi\delta(w)(jw) + \pi\delta(w)(jw)^2 + 1 + jw + 1\right)$$

$$= e^{-3jw}\left(\pi jw\,\delta(w) + \pi(jw)^2\delta(w) + jw + 2\right)$$

$$W(jw) = H_{eq}(jw)\,X(jw)$$

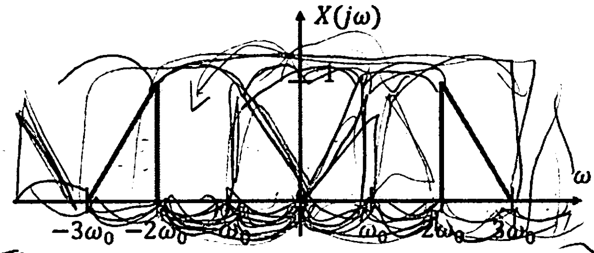
$$A = |H(jw)|$$

$$\theta = \angle H(jw)$$

$$w(t) = |H(jw)|\,e^{j\angle H(jw)}\,x(t)$$

Input: e^{j} [illegible]

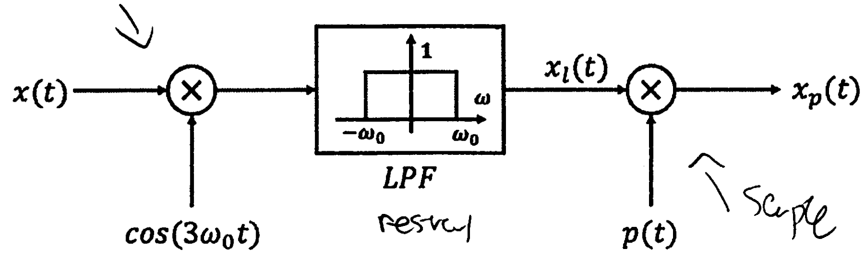
3. **Sampling and Modulation** (40 points)
Assume we have a continuous bandpass signal $x(t)$ with frequency spectrum as shown below. We also assume that $x(t)$ is real. The sampling theorem states that, to recover a signal without distortion, a signal must be sampled at a rate greater than twice its bandwidth. However, since $x(t)$ has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than the Nyquist rate could be used. Now consider the system shown below where $p(t)$ is the sampling function.



$B = 4(2)$

$2 \sim 2\pi B$

modulate



(a) (5 points) What is the Nyquist rate of $x(t)$?

$2\pi B = 3w_0$

$B = \frac{3w_0}{2\pi}$

$\boxed{2B = \frac{3w_0}{\pi}}$

$\frac{2\pi}{T_0}$

$F = \frac{1}{T}$

$2B = \frac{1}{T}$

$T = \frac{1}{2B}$

$T =$

$\frac{2w_0}{\pi}$

(b) (5 points) What is the Nyquist rate of $x_l(t)$? Sketch the frequency spectrum after the low pass filter, i.e. $X_l(j\omega)$.

$$F(x(t)\cos(3w_o t))$$

$$\frac{1}{2}[X(j(w-3w_o)) + X(j(w+3w_o))]$$

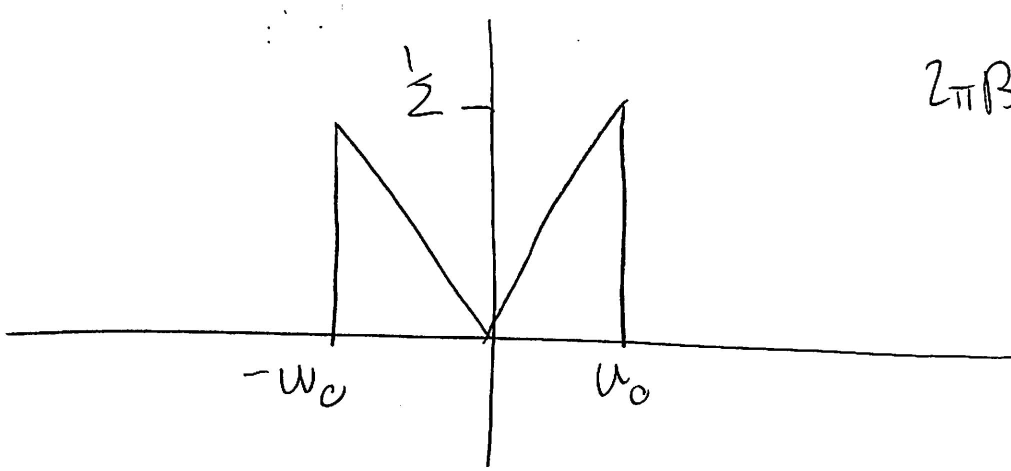
NR: $w_o = 2\pi B$

$$\frac{3w_o}{2\pi} = B$$

$$\boxed{\frac{w_o}{\pi} = 2B}$$

$F = \frac{1}{T}$ $T = \frac{2\pi}{B}$

$2\pi B = 2w_o$



(c) (10 points) If the sampling function is an impulse train

$$p(t) = \sum_{k=-\infty}^{k=+\infty} \delta(t - kT)$$

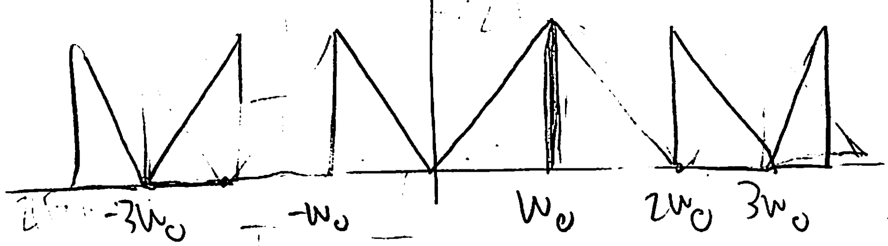
find the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$. Sketch the output ~~frequency spectrum~~ $X_p(jw)$.

$3W_0$

$f(W_0)$

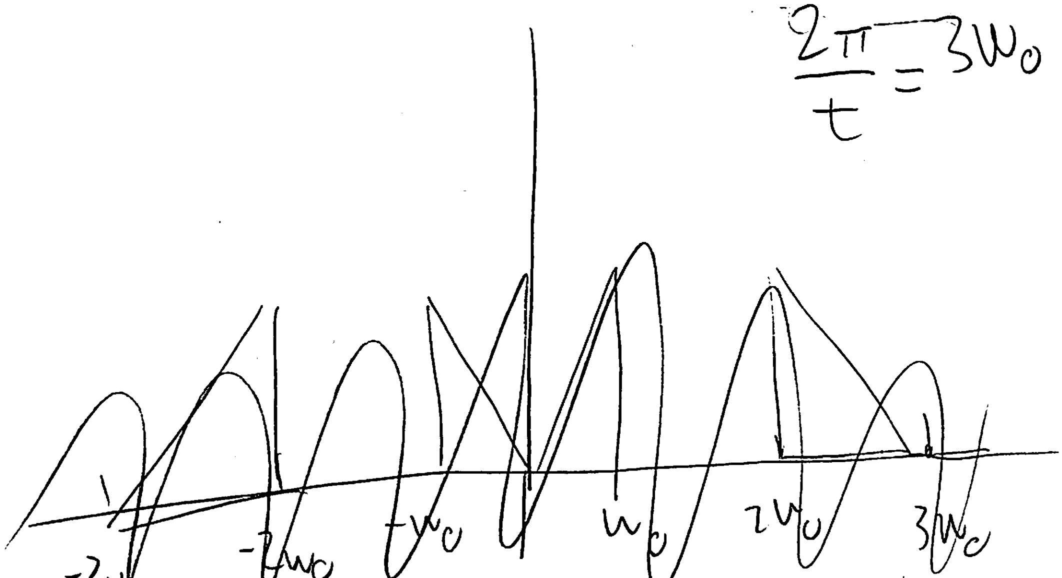
$\delta(t) \leftrightarrow W_0$

$\delta_t(t) \leftrightarrow \frac{2\pi}{t}\delta_{\frac{2\pi}{t}}(w)$

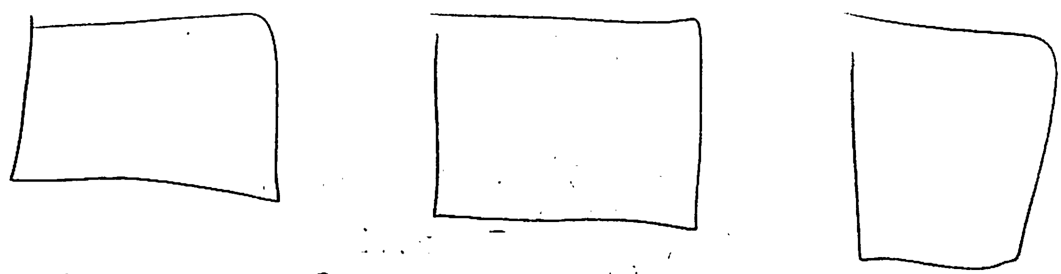


$\frac{W_0}{\pi}$ $\frac{\frac{2\pi}{T}}{\pi}$ $\boxed{T = \frac{2\pi}{3W_0}}$ $\frac{2\pi}{\frac{2}{3}\frac{\pi}{W_0}}$

$\frac{2\pi}{t} = 3W_0$



(d) (20 points) With the $p(t)$ found in part (c), design a system to recover $x(t)$ from $x_p(t)$ without using a bandpass or highpass filter. Note that the recovered signal should have the same amplitude as $x(t)$ in frequency spectrum. Draw a flow diagram of your system and clearly state each component (including cutoff frequencies of any lowpass filter). Write out the explicit mathematical expression of any signal involved.



- Scaling by 2 to get rid of modulation

- multiply by cos then lowpass

4. **Laplace Transform** (30 points)

A system can be described by the following differential equation:

$$y''(t) + y'(t) - 2y(t) = 6x'(t) - 3x(t)$$

where the initial conditions are all zero, i.e. $y''(0) = 0$, $y'(0) = 0$ and $y(0) = 0$.

(a) (10 points) Find the transfer function $H(s) = Y(s)/X(s)$. Assume $x(0) = 0$.

$$s^2 Y(s) + s Y(s) - 2Y(s) = 6s X(s) - 3 X(s)$$

$$Y(s)(s^2 + s - 2) = X(s)(6s - 3)$$

$$\boxed{H(s) = \frac{6s-3}{s^2+s-2}}$$

(b) (20 points) If the input is

$$x(t) = e^{-t}u(t) \leftrightarrow \frac{1}{s+1}$$

then find the output $y(t)$.

$$Y(s) = \frac{X(s)(6s-3)}{s^2+s-2}$$

$$\mathcal{L}^{-1}(Y(s)) = \frac{6s-3}{(s+1)(s+2)(s-1)}$$

$$\frac{6s-3}{(s+1)(s+2)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s-1)}$$

$$\frac{6s-3}{(s+2)(s-1)} = A\Big|_{s=-1} \qquad \frac{6s-3}{(s+1)(s-1)} = B\Big|_{s=-2} \qquad \frac{6s-3}{(s+1)(s+2)} = C\Big|_{s=1}$$

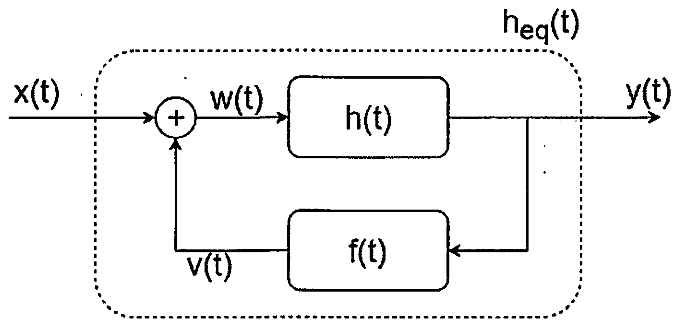
$$\frac{-9}{-2} = A \qquad \frac{-15}{3} = B \qquad \frac{3}{6} = C$$

$$\frac{9}{2} = A \qquad B = -5 \qquad \frac{1}{2} = C$$

$$Y(s) = \frac{9}{2}\left(\frac{1}{s+1}\right) + \frac{-5}{s+2} + \frac{1}{2}\frac{1}{(s-1)}$$

$$\boxed{y(t) = \frac{9}{2}u(t)e^{-t} - 5u(t)e^{-2t} + \frac{1}{2}e^{t}}$$

5. **Feedback System** (45 points)
Consider the feedback system shown below (all components are LTI):



where $h(t) = e^{-2t}u(t)$ and $y(0) = 0$.

(a) (10 points) Show that

$$H_{eq}(s) = \frac{H(s)}{1 - H(s)F(s)}$$

$w(t) = x(t) + v(t)$

$W(s) = X(s) + V(s)$

$Y(s) = H_{eq}(s)\, X(s)$

$y(t) = h(t) * w(t)$

$V(s) = F(s)\, Y(s)$

$Y(s) = H(s)\, W(s)$

$Y(s) = H(s)\, (X(s) + F(s)Y(s))$

$Y(s) = H(s)X(s) + H(s)F(s)Y(s)$

$$Y(s) = \frac{H(s)X(s)}{1 - H(s)F(s)}$$

$$H_{eq} = \boxed{\frac{Y(s)}{X(s)} = \frac{H(s)}{1 - H(s)F(s)}}$$

(b) (10 points) Find the Laplace Transform $H(s)$ of $h(t)$. What is the frequency response $H(j\omega)$? Why is this a low-pass filter?

$$h(t) = e^{-2t}u(t)$$

$$H(s) = \frac{1}{s+2}$$ ROC $Re(s) > -2$, contains $j\omega$

$$H(j\omega) = \frac{1}{j\omega+2}$$

As $\omega \to \infty$, $H(j\omega) \to 0$

As $\omega \to 0$, $H(j\omega) \to \frac{1}{2}$

$sY(s) - y(0)$

(c) (10 points) $v(t)$ and $y(t)$ satisfy the differential equation

$$v(t) = \frac{d}{dt}y(t) + y(t) - 10\int_0^t y(\tau)d\tau$$

What is $F(s)$?

$$V(s) = F(s)\,Y(s)$$

$$\frac{V(s)}{Y(s)} = F(s)$$

$$V(s) = sY(s) - \cancel{y(0)}^{\,0} + Y(s) - 10\,\frac{Y(s)}{s}$$

$$V(s) = Y(s)\left(s + 1 - \frac{10}{s}\right)$$

$$\boxed{H(s) = \frac{V(s)}{Y(s)} = s + 1 - \frac{10}{s}}$$

=

(d) (15 points) Using $F(s)$ found in part c, what is $h_{eq}(t)$? Is this a low-pass, band-pass, or high-pass filter?

$$H(s) = s + 1 - \frac{10}{s}$$

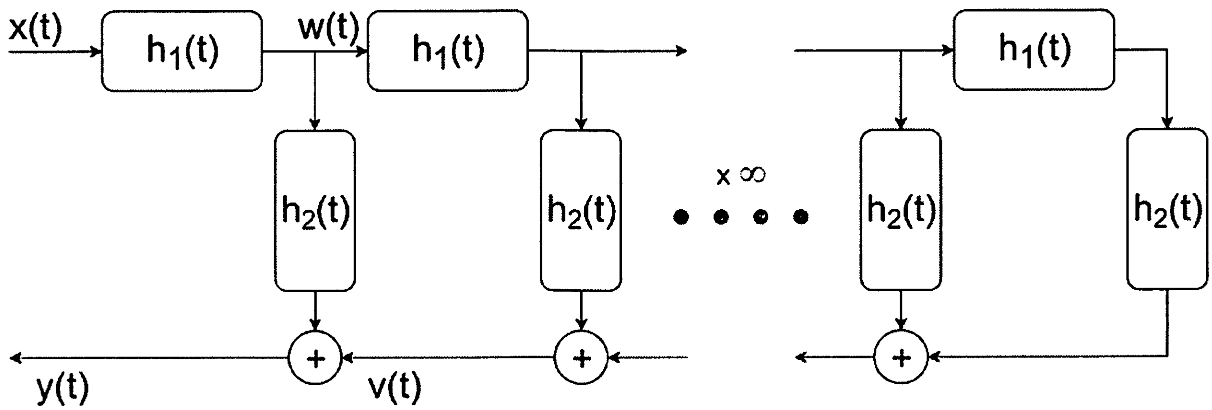
$$h(t) = \delta'(t) + \delta(t) - 10u(t)$$

$$H(j\omega) = j\omega + 1 - 10\pi\delta(\omega) + \frac{1}{j\omega}$$

$$= \frac{j\omega^2 + j\omega - 10\pi j\omega\,\delta(\omega)}{j\omega}$$

As $\omega \to \infty$

Bonus (15 points) Consider the LTI system S shown below, which is a system ladder with an infinite number of rungs. Let $y(t) = S[x(t)]$.



(a) (8 points) In terms of $H_1(s)$ and $H_2(s)$, what is the equivalent transfer function $H_{eq}(s)$ between $Y(s)$ and $X(s)$? *Hint: how does* $\frac{V(s)}{W(s)}$ *relate to* $\frac{Y(s)}{X(s)}$*?*

$Y(s) = H_2(s) + V(s)$

(b) (7 points) Suppose $h_1(t) = e^{-a_1 t}u(t)$ and $h_2(t) = e^{-a_2 t}u(t)$, where a_1 and a_2 are real and positive. For what values of a_1 is S BIBO stable?