ECE 102 Final

TOTAL POINTS

202 / 210

QUESTION 1

1 25 pts

1.1 a **8 / 8**

✓ **- 0 pts** Correct

- 2 pts Partially correct/complete answer

- 6 pts Wrong approach

- 4 pts Wrong use of Fourier transform properties

- 8 pts Wrong answer

1.2 b **10 / 10**

✓ **- 0 pts** Correct

- 8 pts Only mentioned the modulation property/Not enough explanation/ Wrong approach

- 9 pts No explanation

- 1 pts Approach is correct, but did not lead to cos(theta) in the final answer (because of the previous part/something was missed)

- 1 pts Missed the 1/2 of the second modulation

- 10 pts No attempt

- 2 pts Partially correct answer.

- 4 pts Found only the Fourier transform of the signal before the low-pass filter.

1.3 c **7 / 7**

✓ **- 0 pts** Correct

- 2 pts Partially correct

- 1 pts wrong scaling

- 5 pts wrong approach

- 7 pts No attempt

QUESTION 2

2 41 pts

2.1 a **14 / 14**

✓ **- 0 pts** Correct

- 2 pts wrong scaling

- 4 pts Did not expand the delta train in the freq domain

- 1 pts Wrong width or scaling for the rect/ wrong scaling for deltas

- 4 pts Express p(t) as Fourier series, but did not find the coefficients.

- 2 pts Did not apply the sampling property correctly/ did not simplify enough.

- 11 pts Incomplete approach

- 2 pts Wrong expression for p(t)

- 14 pts No attempt.

2.2 b **10 / 10**

✓ **- 0 pts** Correct

- 2 pts Scaling of the convolution is missed

- 4 pts Incomplete answer

- 4 pts Did not simplify/ did not apply sampling property before doing convolution

- 10 pts No attempt

- 8 pts Wrong answer

- 2 pts sinc is missed

- 2 pts Did not apply properly the shifting property

- 8 pts Only stated the convolution

2.3 c **10 / 10**

✓ **- 0 pts** Correct

- 4 pts The limits of the shifted triangles is wrong (on the w a-xis)

- 2 pts Wrong scaling

- 10 pts No attempt

- 4 pts Partially correct graphical interpretation of X(w)

- 8 pts Wrong answer

- 3 pts Missing the central triangle

- 3 pts Partially correct sketch because of the previous wrong part (convolution with a sinc)

2.4 d 7 / 7

✓ **- 0 pts** Correct

- 4 pts Gave only an expression of Y(jw) as multiplication of rect and X(jw) / Gave an explanation of it should be done/ Partially correct answer

- 7 pts No attempt/ Wrong answer

- 5 pts Incomplete answer

- 2 pts Wrong scaling

QUESTION 3

3 30 pts

3.1 a 15 / 15

✓ **+ 15 pts** Correct

+ 0 pts Wrong/no answer

+ 6 pts Wrong answer because of using LTI property for non LTI system, partial credit for cascade systems equations

+ 7 pts Wrong property for scaling of convolution, partial credit for right path toward solution

- 1 pts Error in scaling factors

- 2 pts Error in scaling of arguments

+ 9 pts Right initial path and correct properties but missing final steps

- 1 pts missing final form of H(jw)

3.2 b 15 / 15

✓ **+ 15 pts** Correct

+ 0 pts Wrong/No answer

- 1 pts missing scale factor of two or wrong power of e^{-4at}, extra scale

- 1.5 pts missing the point that \delta(bt)=1/b\delta(t) , u(bt)=u(t) for b>0

+ 10 pts correct h(t) but wrong heq(t) due to wrong answer from previous parts

+ 9 pts derived h(t) but no heq(t)

- 2 pts right answer but missing simplifications/errors for heq(t)

+ 5 pts right initial steps

- 3 pts mistakes in inverse Fourier transform

QUESTION 4

4 40 pts

4.1 a 10 / 10

✓ **+ 5 pts** **correct form of input/output relation as first step**

✓ **+ 5 pts** **correct form of H1(s) as fraction of Y(s)/V(s) and then express as terms of H(s)**

+ 2 pts correct equation of Y(s)=V(s)H1(s)

+ 0 pts Wrong/No answer

4.2 b 5 / 5

✓ **+ 5 pts** Correct

+ 2 pts Partial credit for correct path toward answer, however final form is missing

+ 0 pts Wrong/ No answer

+ 1 pts partial credit for right equation of H=Y(s)/X(s) or Heq(s)=G(s)G1(s)

4.3 c 10 / 10

✓ **+ 10 pts** **Correct, using the correct Heq(s) leading to right form of H(s).**

+ 2 pts Credit for right form of Heq(s)

+ 0 pts Wrong/ no answer

- 1 pts Correct answer, however minor error in negative sign

+ 7 pts Partial credit, path toward answer is correct but since the answer from previous part is not correct, the final answer is not the right form

- 1.5 pts Right answer but missing the final form of H(s) in terms of G(s)

4.4 d 15 / 15

✓ **+ 15 pts** Correct

+ 0 pts Wrong/ No answer

+ 12 pts correct solution, however due to a mistake in negative sign from previous part the final solution is incorrect

+ 8 pts Error in [Laplace transform / plugging in G(s) / in simplifying the H(s)] partial credit for right path toward answer

+ 3 pts credit for right form of G(s)

- **3 pts** missing/wrong the inverse Laplace transform
- **1.5 pts** minor error in inverse Laplace transform

QUESTION 5

5 5 20 / 20

✓ + **6 pts** **Laplace transform correct.**

✓ + **6 pts** **Input Laplace transform correct.**

✓ + **6 pts** **Residue calculation correct.**

✓ + **2 pts** **Inversion of Laplace transform correct. (I did not take off points if you forgot to put back in the exp(-8) term)**

+ **5 pts** LAPLACE ERROR: sign or division mistake.

+ **4 pts** LAPLACE ERROR: Did not take Laplace transform of RHS.

+ **2 pts** LAPLACE ERROR: Did not take the Laplace transform of all terms.

+ **4 pts** INPUT ERROR: Neglected e^{-8} term or other similar mistake.

+ **2 pts** INPUT ERROR: no time shift or other substantial error.

+ **5 pts** RESIDUE ERROR: minor algebra

+ **3 pts** RESIDUE ERROR: Factored poles incorrectly

+ **3 pts** RESIDUE ERROR: did not solve for the residues correctly.

+ **3 pts** RESIDUE ERROR: did not include s+5

+ **1 pts** INVERSION ERROR: did not time shift correctly or other minor mistake.

+ **0 pts** No attempt or incorrect.

QUESTION 6

6 44 pts

6.1 a.i 3 / 6

+ **6 pts** Correct (False) with appropriate justification (e.g., convolution of constant w/ odd function; or two non-overlapping Fourier Transforms)

+ **4 pts** Correct (False) with a counter example that has a modest mistake in it (usually doing convolution wrong) but the correct intuition.

+ **3 pts** Correct (False) with a counter example that has a substantial mistake in it or a non-rigorous statement (e.g., not enough to say the integral can be zero if x and y are nonzero).

✓ + **3 pts** **Incorrect (True) with a Fourier Transform explanation, but not realizing that it's possible to get X(jw)*Y(jw) = 0 w/ e.g., two non-overlapping rects (as in a HW example).**

+ **2 pts** Incorrect (True) by arguing that the convolution formula implies one of them has to be zero; but not accounting for how the integral of something non-zero can be zero; or another related convolution intuition (e.g., flip and drag)

+ **0 pts** Incorrect (True) without justification (or inadequate justification), or no attempt.

6.2 a.ii 6 / 6

✓ + **6 pts** **Correct (false) with appropriate counterexample. Since our question stated any counter example was sufficient, we did give full points for the trivial counterexample. But we wanted you to see that e.g., sinc*sinc = sinc (since it's two rects multiplying in the frequency domain).**

+ **3 pts** Incorrect (true) by taking the Fourier Transform but not identifying that e.g., a bandlimited signal could be multiplied by a bandlimited rect (as opposed to delta), giving the same result.

+ **3 pts** Incorrect (true) by stating the sifting property or identity property of convolution.

+ **2 pts** Correct (false) but with an incorrect justification

+ **0 pts** Incorrect with inadequate or no justification, or no attempt

6.3 a.iii 6 / 6

✓ + **6 pts** **Correct with adequate justification.**

+ **5 pts** Did work mostly correct but came up with an incorrect Nyquist rate (like 3B, 4B, 8B, 10B, or 12B, due to minor algebra mistake)

+ **3 pts** Expressed the FT of the signal correctly3, but did not infer Nyquist correctly (through e.g., not realizing what the largest freq component is or making an incorrect argument about the modulated replicas or arguing the signal is no longer bandlimited due to the phase term).

+ **3 pts** Drew diagram (or equivalent) incorrectly leading to an incorrect Nyquist rate.

+ **2 pts** Recognized the Nyquist of x(2t) is 4B, but made an argument that the cosine term did not matter or was zero (or entirely neglected it).

+ **1 pts** Recognized Nyquist of x(t) is 2B; or other preliminary work.

+ **0 pts** No attempt or either correct or incorrect without appropriate justification.

6.4 a.iv **6 / 6**

✓ + **6 pts** **Correct with appropriate justification. (I also gave credit if you were off by 2*pi; remember those factors in the future.)**

+ **6 pts** Correct w/ an intuitive argument stating Parseval's theorem and realizing the rect would be compressed by a factor of 3.

+ **4 pts** Algebra mistake (e.g., a very common mistake was not removing the phase term in the Parseval integral; remember, it's magnitude squared, so you multiply by the complex conjugate. Also intuitively, the magnitude of something should not be complex.)

+ **3 pts** Correct calculations in the frequency domain but did not use Parseval's theorem or evaluate the Parseval integral fully.

+ **2 pts** Correct but without a rigorous argument.

+ **2 pts** Attempted to write out energy integral, but either did not solve the integral completely or made incorrect conclusions about the integral.

+ **0 pts** No attempt or incorrect w/o appropriate work.

6.5 b **10 / 10**

✓ + **10 pts** **Correct proof**

+ **9.5 pts** Essentially correct proof but believed the statement was false due to a factor of 2*pi (the factor of 2pi is in the frequency domain convolution theorem).

+ **6 pts** Proof uses invalid or incorrect steps.

+ **5 pts** Attempted to use the integral theorem (this is not a running integral) and does not use appropriate steps to come to the identity.

+ **3 pts** Statement of convolution theorem (or attempt to evaulate convolution) and definition of Fourier transform, but did not plug in omega=0.

+ **0 pts** Not attempted or does not have appropriate justification (irrespective of if correct or incorrect).

6.6 C **10 / 10**

✓ + **5 pts** **Correct (stable) with appropriate justification.**

✓ + **5 pts** **Correct (not causal) with appropriate justification.**

+ **3 pts** Correct (not causal) but did not correctly time-shift/reverse the rect.

+ **3 pts** Incorrect (causal) but with correct work.

+ **2 pts** Incorrect (unstable) due to trying to evaluate absolute integral of impulse response but making a mistake (like not changing the bounds of integration or changing the bounds incorrectly); or by taking the Laplace transform, but note, this is not a causal signal so we can't take the unilateral Laplace transform.

+ **2 pts** Correct (stable) with an explanation in the spirit of what's correct but not actually correct.

+ **2 pts** Corect (causal) with an explanation in the spirit of what's correct but not actually correct.

+ **2 pts** Points for drawing the rect correctly (absent other work or conclusions about stability and causality; or with an incorrect conclusion on causality)

+ **0 pts** No attempt or no appropriate justification on both counts (i.e., stability and causality).

+ **0 pts** Incorrect causality

QUESTION 7

7-Bonus 10 pts

7.1 a **0 / 5**

- **0 pts** Correct

- **2 pts** Partially correct approach

✓ - **5 pts** **Wrong Approach (Fourier cannot be used here, h2(t) is not causal)/ No attempt**

- **4 pts** Incomplete Approach

7.2 b **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Right frequencies limits for each term, but wrong nyquist rate

- 5 pts Wrong Answer/ No attempt

- 3 pts Not precise explanation

- 2 pts Wrong computation of freq components

ECE 102, Fall 2018
Department of Electrical and Computer Engineering
University of California, Los Angeles

Final Exam
Prof. J.C. Kao
TAs: H. Salami, S. Shahsavari

UCLA True Bruin academic integrity principles apply.
Open: Four pages of cheat sheet allowed.
Closed: Book, computer, internet.
11:30am-2:30pm, Haines Room 118
Tuesday, 11 Dec 2018.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

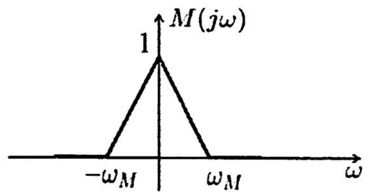
Signature: ____________________

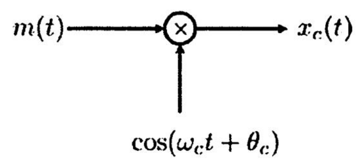
ID#: ________________

Problem 1	_____	/ 25
Problem 2	_____	/ 41
Problem 3	_____	/ 30
Problem 4	_____	/ 40
Problem 5	_____	/ 20
Problem 6	_____	/ 44
BONUS	_____	/ 10 bonus points
Total	_____	/ 200 points + 10 bonus points

Problem 1 (25 points)

Consider a bandlimited signal $m(t)$, its frequency spectrum $M(j\omega)$ is shown below. We modulate $m(t)$ with $\cos(\omega_c t + \theta_c)$, where θ_c is a constant phase but unknown:





(a) (8 points) Express $X_c(j\omega)$, the Fourier transform of $x_c(t)$, in terms of $M(j\omega)$. *Hint: use the fact that* $\cos(u) = \dfrac{e^{ju} + e^{-ju}}{2}$.

$$x_c(t) = m(t)\cos(\omega_c t + \theta_c)$$

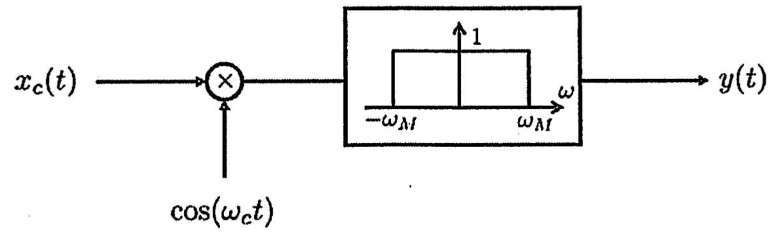
$$= \frac{m(t)}{2}\left(e^{j(\omega_c t + \theta_c)} + e^{-j(\omega_c t + \theta_c)}\right)$$

$$= \frac{1}{2} m(t)\left(e^{j\theta_c} e^{j\omega_c t} + e^{-j\theta_c} e^{-j\omega_c t}\right)$$

$$X_c(j\omega) = \frac{1}{2} M(j\omega) * \left(e^{j\theta_c}\,\delta(\omega - \omega_c) + e^{-j\theta_c}\,\delta(\omega + \omega_c)\right)$$

$$= \boxed{\frac{1}{2}\left(e^{j\theta_c} M(j(\omega - \omega_c)) + e^{-j\theta_c} M(j(\omega + \omega_c))\right)}$$

(b) (10 points) We demodulate $x_c(t)$ as follows:



Show that $y(t) = \dfrac{1}{2}\cos(\theta_c)m(t)$. Assume $\omega_c \gg \omega_M$.

$$
\begin{aligned}
Y(j\omega) &= \mathcal{F}\{x_c(t)\cos(\omega_c t)\}(\omega)\cdot \mathrm{rect}\left(\frac{\omega}{2\omega_m}\right)\\
&= \frac{1}{2}\left(X_c(j(\omega-\omega_c)) + X_c(j(\omega+\omega_c))\right)\ \mathrm{rect}\left(\frac{\omega}{2\omega_m}\right)\\
&= \frac{1}{4}\left(e^{j\vartheta_c} M(j(\omega-2\omega_c)) + e^{-j\vartheta_c} M(j\omega)\right.\\
&\qquad \left. + e^{j\vartheta_c} M(j\omega) + e^{-j\vartheta_c} M(j(\omega+2\omega_c))\right)\ \mathrm{rect}\left(\frac{\omega}{2\omega_M}\right)\\
&= \frac{1}{4}\left(e^{-j\vartheta_c} M(j\omega) + e^{j\vartheta_c} M(j\omega)\right)\\
&= \frac{1}{2} M(j\omega)\left(\frac{e^{-j\vartheta_c} + e^{j\vartheta_c}}{2}\right)\\
&= \frac{1}{2} M(j\omega)\underbrace{\cos(\vartheta_c)}_{\text{constant}}.
\end{aligned}
$$

$$
y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}(t) = \frac{1}{2}\cos(\vartheta_c)\, m(t). \qquad \square
$$

(c) (7 points) Assume that you also know $z(t) = \frac{1}{2}\sin(\theta_c)m(t)$. How can you recover $m(t)$ from $y(t)$ and $z(t)$?
Hint: $\cos^2(u) + \sin^2(u) = 1$.

$$y^2(t) = \frac{1}{4}\cos^2(\vartheta_c)\, m^2(t)$$
$$z^2(t) = \frac{1}{4}\sin^2(\vartheta_c)\, m^2(t)$$

$$y^2(t) + z^2(t) = \frac{1}{4} m^2(t).$$

$$\sqrt{y^2(t)+z^2(t)} = \frac{1}{2}|m(t)| = \frac{1}{2}m(t).$$

$$\boxed{m(t) = 2\sqrt{y^2(t)+z^2(t)}}$$

Note: $M(j\omega) = \Delta\left(\frac{\omega}{\omega_m}\right)$ from diagram

$m(t) = \mathcal{F}^{-1}\{\Delta(\frac{\omega}{\omega_m})\}(t)$

and is of the form $k\,\text{sinc}^2(ct)$

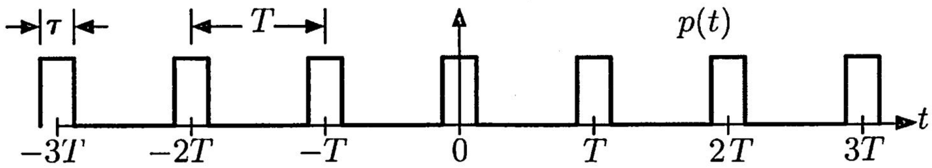
where k is a positive constant. and c is another constant.

As such, we could ensure $m(t) \geq 0$,

and $|m(t)| = m(t)$.

Problem 2 (41 points)

Consider the following sequence of short rect($\cdot$) pulses, denoted by $p(t)$:



Each rect($\cdot$) pulse has width τ, and the pulses are spaced by T as diagrammed above.

(a) (14 points) Find $P(j\omega)$, the Fourier transform of $p(t)$. Express $P(j\omega)$ as a sum, and simplify where possible. *Hint: One approach is to write $p(t)$ as convolution of a rect function with an impulse train.*

$$p(t) = \text{rect}\left(\frac{t}{\tau}\right) * \delta_T(t).$$

$$P(j\omega) = \mathcal{F}\left\{\text{rect}\left(\frac{t}{\tau}\right)\right\}(\omega) \cdot \omega_0 \delta_{\omega_0}(\omega) \qquad \omega_0 = \frac{2\pi}{T}$$

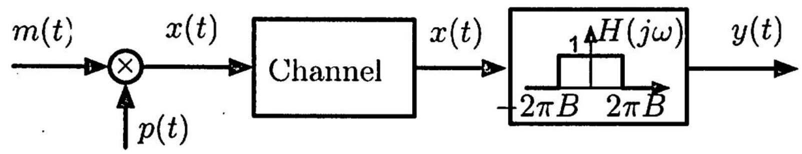
$$= \tau \,\text{Sinc}\left(\frac{\omega\tau}{2\pi}\right) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$= \frac{2\pi\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

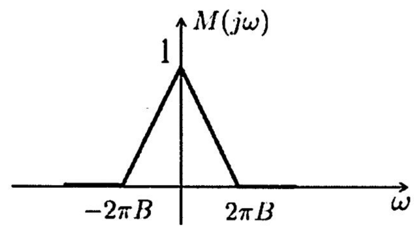
$$= \frac{\cancel{2\pi}\cancel{\tau}}{\cancel{T}} \cdot \frac{2}{\cancel{\tau}} \cdot \sum_{k=-\infty}^{\infty} \frac{\cancel{T}}{\cancel{2\pi}k} \sin\left(\frac{2\pi k\tau}{2T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$= \boxed{2 \sum_{k=-\infty}^{\infty} \frac{1}{k} \sin\left(\frac{\pi k\tau}{T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)}$$

(b) (10 points) Consider the following system:



where the input $m(t)$ is multiplied with the rect pulse train, $p(t)$. The signal $m(t)$ is bandlimited and it has the following frequency spectrum:



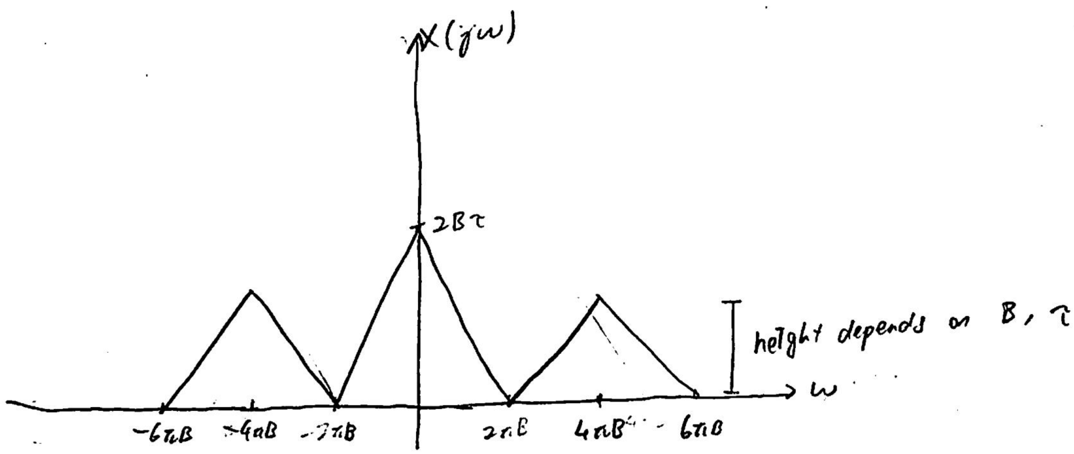
Assume that the rect$(\cdot)$ pulses are spaced by $T = \frac{1}{2B}$. Express the spectrum $X(j\omega)$ of $x(t)$ in terms of $M(j\omega)$.

$$P(j\omega) = 2 \sum_{k=-\infty}^{\infty} \frac{1}{k} \sin(2B\pi k \tau)\, \delta(\omega - 4B\pi k)$$

$$X(j\omega) = \frac{1}{2\pi} (M * P)(j\omega)$$

$$= \boxed{\frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k} \sin(2B\pi k \tau)\, M(j(\omega - 4B\pi k))}$$

(c) (10 points) Sketch $X(j\omega)$ for $-6\pi B \leq \omega \leq 6\pi B$.



(d) (7 points) Find the spectrum of the signal at the output of the lowpass filter $Y(j\omega)$, i.e., find an expression of $Y(j\omega)$ in terms of $M(j\omega)$.

$$Y(j\omega) = X(j\omega)\, rect\left(\frac{\omega}{4\pi B}\right)$$

$$= \lim_{k \to 0} \frac{1}{\pi} \frac{\sin(2B\pi k\tau)}{k} M(j\omega)$$

← all other terms in infinite sum vanished due to $rect\left(\frac{\omega}{4\pi B}\right)$

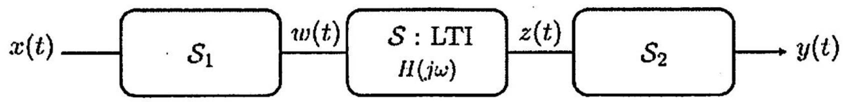
$$= \frac{1}{\pi} M(j\omega) \lim_{k \to 0} \frac{\sin(2B\pi k\tau)}{k}$$

$$= \frac{1}{\pi} M(j\omega) \lim_{k \to 0} 2B\pi\tau \underbrace{\cos(2B\pi k\tau)}_{=1 \text{ when } k=0}$$

$$= \boxed{2B\tau M(j\omega)}$$

Problem 3 (30 points)

An LTI system S is cascaded in series with two other non-LTI systems as follows:



The system S_1 is given by:

$$w(t) = x\left(\frac{t}{2}\right)$$

And the system S_2 is:

$$y(t) = z(2t)$$

The system S has $H(j\omega)$ as its frequency response.

(This question continues on the next page.)

(a) (15 points) Find how $Y(j\omega)$ is related to $X(j\omega)$, in terms of $H(j\omega)$. Deduce the overall frequency response $H_{eq}(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$.

$W(j\omega) = 2X(2j\omega)$ by time scaling.

$z(t) = (w * h)(t) \Rightarrow Z(j\omega) = W(j\omega)H(j\omega) = 2X(2j\omega)H(j\omega)$.

$Y(j\omega) = \frac{1}{2} Z\left(\frac{j\omega}{2}\right) = X(j\omega)H\left(\frac{j\omega}{2}\right)$.

$$H_{eq}(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \boxed{H\left(\frac{j\omega}{2}\right)}$$

(b) (15 points) If $H(j\omega)$ is given by:

$$H(j\omega) = \frac{2a - j\omega}{2a + j\omega}$$

where $a > 0$, find the impulse response $h(t)$ of the system $\mathcal{S}$. Deduce the overall impulse response $h_{eq}(t)$.

$$H(j\omega) = \frac{2a}{2a+j\omega} - \frac{j\omega}{2a+j\omega} = 2a\left(\frac{1}{2a+j\omega}\right) - \omega\left(\frac{1}{\frac{2a}{j}+\omega}\right)$$

$$= 2a\left(\frac{1}{2a+j\omega}\right) - \omega\left(\frac{1}{\omega-2aj}\right) = 2a$$

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}(t)$$

$$= 2ae^{-2at}u(t) - \omega e^{-\omega}$$

$$H(j\omega) = -\frac{-2a+j\omega}{2a+j\omega} = -\frac{2a+j\omega-4a}{2a+j\omega} = \frac{4a}{2a+j\omega} - 1.$$

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}(t)$$

$$= \boxed{4a\,e^{-2at}u(t) - \delta(t).}$$

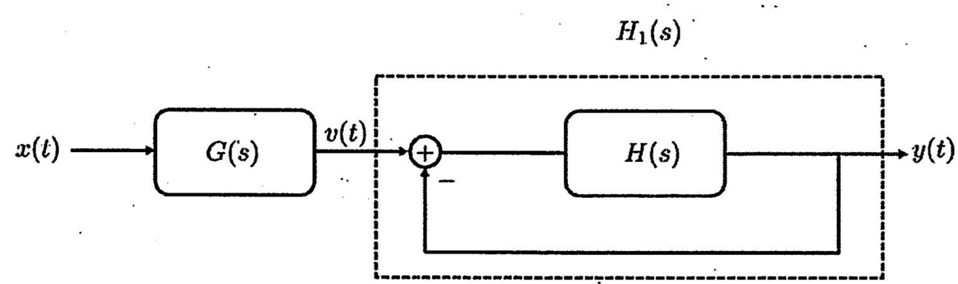
$$H_{eq}(j\omega) = H\left(\frac{j\omega}{2}\right) \Leftrightarrow h_{eq}(t) = 2h(2t).$$

$$= 8a\,e^{-4at}u(2t) - 2\delta(2t)$$

$$= \boxed{8a\,e^{-4at}u(t) - \delta(t)}$$

Problem 4 (40 points)

Consider the following system:



(a) (10 points) Find the transfer function $H_1(s)$ of the system that maps $v(t)$ to $y(t)$.

$$y(t) = ((v - y) * h)(t).$$

In s-domain:

$$Y(s) = (V(s) - Y(s))(H(s))$$

$$Y(s) = V(s)H(s) - Y(s)H(s).$$

$$Y(s)(1 + H(s)) = V(s)H(s)$$

$$H_1(s) = \frac{Y(s)}{V(s)} = \boxed{\frac{H(s)}{1 + H(s)}}$$

Check:

$$\left(V(s) - V(s)\left(\frac{H(s)}{1+H(s)}\right)\right)H(s)$$

$$= \left(V(s)\underbrace{\left(1 - \frac{H(s)}{1+H(s)}\right)}_{\frac{1}{1+H(s)}}\right)H(s)$$

$$= V(s)\frac{H(s)}{1+H(s)} = Y(s) \checkmark$$

(b) (5 points) Find the overall transfer function $H_{eq}(s)$.

$$Y(s) = X(s)\ G(s)\ H_1(s)$$

$$H_{eq}(s) = \frac{Y(s)}{X(s)} = G(s)\ H_1(s) = \boxed{\frac{G(s)\,H(s)}{1 + H(s)}}$$

(c) (10 points) How can we choose $H(s)$ in terms of $G(s)$ so that the overall system has the following impulse response $h_{eq}(t) = \delta(t)$?

$$H_{eq}(s) = \mathcal{L}\{\delta(t)\}(s) = 1 =$$

$$\frac{G(s)\,H(s)}{1+H(s)} = 1$$

$$G(s)\,H(s) = 1 + H(s)$$

$$G(s) = \frac{1}{H(s)} + 1$$

$$G(s) - 1 = \frac{1}{H(s)}$$

$$H(s) = \boxed{\frac{1}{G(s)-1}}$$

Check: $$G(s)\,\frac{H(s)}{1+H(s)} = \frac{G(s)}{G(s)-1}\;\frac{1}{\frac{G(s)\cancel{-1+1}}{G(s)-1}} = \frac{G(s)}{\cancel{G(s)-1}}\;\frac{\cancel{G(s)-1}}{G(s)} = 1. \checkmark$$

(d) (15 points) Using the relation you found in part (c), find $h(t)$ if $g(t) = e^{-2t}u(t)$.

$$G(s) = \frac{1}{s+2}.$$

$$H(s) = \frac{1}{G(s) - 1} = \frac{1}{\frac{1}{s+2} - \frac{s+2}{s+2}} = \frac{s+2}{-s-1}$$

$$= -\frac{s+2}{s+1} = -1 - \frac{1}{s+1}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}(t) = \boxed{-\delta(t) - e^{-t}u(t)}$$

Problem 5 (20 points)

A system is described by the following differential equation:

$$y''(t) + 5y'(t) + 6y(t) = x'(t) + 5x(t)$$

If the input is

$$x(t) = e^{-4t}u(t-2)$$

find the output $y(t)$. Assume all initial conditions are zero.

There is additional space on the next page if needed.

$$x(t) = e^{-4(t-2)} e^{-8} u(t-2)$$

$$X(s) = e^{-8} e^{-2s} \mathcal{L}\{e^{-4t} u(t)\}(s)$$

$$= \frac{e^{-2s-8}}{s+4}$$

$$\cancel{X(s) = \frac{e^{-2(s+4)}}{s+4}}$$

$$\cancel{X(s-4) = \frac{e^{-2s}}{s}}$$

$$\cancel{\mathcal{L}^{-1}\{X(s-4)\} = u(t-2)}$$

$$\cancel{\mathcal{L}^{-1}\{X(s)\} = e^{-4t}u(t-2)}$$

$$s^2 Y(s) + 5s Y(s) + 6Y(s) = sX(s) + 5X(s)$$

$$Y(s) = X(s)\frac{(s+5)}{s^2+5s+6} = \frac{e^{-2s-8}}{s+4} \cdot \frac{s+5}{(s+2)(s+3)}$$

Let $\frac{s+5}{(s+4)(s+2)(s+3)} = \frac{r_1}{s+2} + \frac{r_2}{s+3} + \frac{r_3}{s+4}$

$$r_1 = \frac{-2+5}{(-2+4)(-2+3)} = \frac{3}{2}. \qquad r_3 = \frac{-4+5}{(-4+2)(-4+3)} = \frac{1}{2}$$

$$r_2 = \frac{-3+5}{(-3+4)(-3+2)} = -2.$$

$$\cancel{r_1(7s+12) + r_2(8s+8) + r_3(5s+6) = \frac{21}{2} - \frac{24}{2} + \frac{5}{2} = 1 = 18-16+3 = 5.}$$

$$Y(s) = e^{-2s-8}\left(\frac{3}{2}\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+3}\right) + \frac{1}{2}\left(\frac{1}{s+4}\right)\right).$$

$$y(t) = e^{-8}\mathcal{L}^{-1}\left\{\frac{3}{2}\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+3}\right) + \frac{1}{2}\left(\frac{1}{s+4}\right)\right\}(t-2)$$

$$= e^{-8}\left(\frac{3}{2}e^{-2(t-2)} - 2e^{-3(t-2)} + \frac{1}{2}e^{-4(t-2)}\right) u(t-2)$$

$$\boxed{= \left(\frac{3}{2}e^{-2t-4} - 2e^{-3t-2} + \frac{1}{2}e^{-4t}\right)u(t-2)}$$

(Additional space for problem 5.)

Problem 6 (44 points)

(a) (24 points) Determine if each of the following four statements is true or false. When the statement is false, a counter example is sufficient. If the statement is true, you must justify your answer to receive full credit.

i. If $x(t) * y(t) = 0$, then $x(t) = 0$ or $y(t) = 0$.

~~$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$~~

~~For this integral to be zero everywhere, the value of t must not be depended upon. As such, either $x(t)$ or $y(t)$ (or both) equals zero. $\square$~~

$\boxed{\text{True}}$

Take the Fourier transform. $X(j\omega) Y(j\omega) = 0 \Rightarrow$ either $X(j\omega) = 0$ or $Y(j\omega) = 0$ or both. $X(j\omega) = 0 \Leftrightarrow x(t) = 0$, $Y(j\omega) = 0 \Leftrightarrow y(t) = 0$

ii. If $x(t) * h(t) = x(t)$, then $h(t)$ must be an impulse, i.e., $h(t) = \delta(t)$.

$\boxed{\text{False.}}$ Let $x(t) = 0$, $h(t) = 0$.

$$(x * h)(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{0} \underbrace{h(t-\tau)}_{0} d\tau = 0 = x(t),$$

but $h(t) \neq \delta(t)$.

iii. A signal $x(t)$ is bandlimited where its Fourier transform $X(j\omega) = 0$ for $|\omega| > 2\pi B$ rad/s. The Nyquist rate of $\cos(4\pi Bt)x(t-2) + x(2t)$ is $6B$ Hz.

$\boxed{\text{True.}}$

$$\mathcal{F}\{\cos(4\pi Bt)\, x(t-2) + x(2t)\}(\omega)$$

$$= \frac{1}{2}\left(\mathcal{F}\{x(t-2)\}(\omega - 4\pi B) + \mathcal{F}\{x(t-2)\}(\omega + 4\pi B)\right) + \frac{1}{2}X\left(j\frac{\omega}{2}\right)$$

$$= \frac{1}{2}\left(e^{-2j(\omega - 4\pi B)} \underbrace{X(j(\omega - 4\pi B))}_{=0\ \forall\ \omega > 6\pi B,\ \omega < 2\pi B} + e^{-2j(\omega + 4\pi B)} \underbrace{X(j(\omega + 4\pi B))}_{=0\ \forall\ \omega > -2\pi B,\ \omega < -6\pi B} + \underbrace{X\left(j\frac{\omega}{2}\right)}_{=0\ \forall\ |\omega| > 4\pi B}\right)$$

Since the signal is bandlimited such that its Fourier transform is zero for $|\omega| > 6\pi B$, $\boxed{6B}$ is its Nyquist rate.

iv. If $x(t) = \text{sinc}(t)$, then the energy of $x(3t+2)$ is $\frac{1}{3}$.

$$\mathcal{F}\{x(t)\}(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\mathcal{F}\{x(t+2)\}(\omega) = e^{2j\omega}\,\text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\mathcal{F}\{x(3t+2)\}(\omega) = \frac{1}{3}e^{\frac{2}{3}j\omega}\,\text{rect}\left(\frac{\omega}{6\pi}\right)$$

$$E = \int_{-\infty}^{\infty} |x(3t+2)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{3}e^{\frac{2}{3}j\omega}\,\text{rect}\left(\frac{\omega}{6\pi}\right)\right|^2 d\omega$$

$$= \frac{1}{9}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\cancel{|e^{2/3 j\omega}|^2}}_{=1}\, \left|\text{rect}\left(\frac{\omega}{6\pi}\right)\right|^2 d\omega$$

$$= \frac{1}{9}\cdot\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} \text{rect}\left(\frac{\omega}{6\pi}\right) d\omega}_{6\pi}$$

$$= \frac{\cancel{6}\,\cancel{\pi}}{\cancel{2\pi}}\cdot\frac{1}{\cancel{9}\,3}$$

$$= \boxed{\frac{1}{3}}$$

true.

(b) (10 points) If $y(t) = x(t) * h(t)$, then show that the following identity holds:

$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} h(t)dt\right) \cdot \left(\int_{-\infty}^{\infty} x(t)dt\right)$$

Hint: One approach is to look at the integral expression for the Fourier transform when $\omega = 0$.

$y(t) = (x * h)(t) \Rightarrow Y(j\omega) = X(j\omega)\, H(j\omega)$.

$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \Rightarrow Y(0) = \int_{-\infty}^{\infty} y(t)\, dt.$ similar for $X(0)$, $H(0)$.

$$\int_{-\infty}^{\infty} y(t)\, dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right)\left(\int_{-\infty}^{\infty} x(t)\, dt\right). \quad \square$$

(c) (10 points) An LTI system has the following impulse response: $h(t) = e^t u(-1-t)$. Is the system stable? Is it causal?

$$h(-2) = e^{-2} \underbrace{u(-1-(-2))}_{=1} = e^{-2} \neq 0 \Rightarrow \text{the system is } \boxed{\text{non-causal.}}$$

Assume $x(t)$ and $y(t)$ is an input-output pair related by $y(t) = (x * h)(t)$. Furthermore, assume $|x(t)| \leq M_x$.

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau \right|$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

$$\leq M_x \int_{-\infty}^{\infty} e^{\tau} u(-1-\tau)\, d\tau$$

$$= M_x \int_{-\infty}^{-1} e^{\tau}\, d\tau.$$

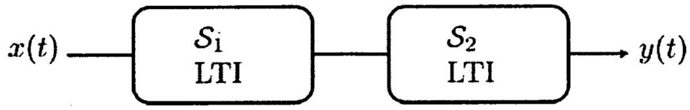
$$= M_x \left(\frac{1}{e}\right)$$

$$= \frac{M_x}{e}.$$

If we let $M_y = \frac{M_x}{e}$, $|y(t)| \leq M_y$. Hence the system is $\boxed{\text{BIBO-stable.}}$

BONUS (10 points)

(a) (5 points) Two LTI systems are linearly cascaded as follows:



The impulse response of the first system is $h_1(t) = e^t u(t)$ and the impulse response of the second system is $h_2(t) = e^{2t}\cos(t)$. What is the impulse response of the equivalent system $h_{eq}(t)$?

We cannot use the Fourier transform as the impulse responses of the systems are not energy signals. ~~However, since $h_1(t)=0\ \forall\ t<0$, $h_{eq}(t)=0\ \forall\ t<0$, and the eq. system is causal. We can use the Laplace transf.~~

$$H_1(s) = \frac{1}{s-1}.$$

$$H_2(s) = \mathcal{L}\{e^{2t}\cos(t)\} = \frac{s-2}{(s-2)^2+1}$$

$$H_{eq}(s) = H_1(s)\,H_2(s) = \frac{s-2}{(s^2-4s+5)(s-1)} = \frac{r_1}{s-1} + \frac{r_{21}s + r_{22}}{(s-2)^2+1}$$

$$r_1 = \left.\frac{s-2}{(s-2)^2+1}\right|_{s=1} = \frac{-1}{1+1} = -\frac{1}{2}.$$

$$r_1\cancel{s^2} + r_{21}\cancel{s^2} = 0$$

$$r_{21} = -r_1 = \frac{1}{2}.$$

$$H_{eq}(0) = \frac{-2}{5\,(-1)} = \frac{-\frac{1}{2}}{0-1} + \frac{r_{22}}{5}$$

$$\frac{2}{5} = \frac{1}{2} + \frac{r_{22}}{5}$$

$$4 = 5 + 2r_{22} \Rightarrow r_{22} = -\frac{1}{2}.$$

$$H_{eq}(s) = -\frac{1}{2(s-1)} + \frac{s-1}{2((s-2)^2+1)}$$

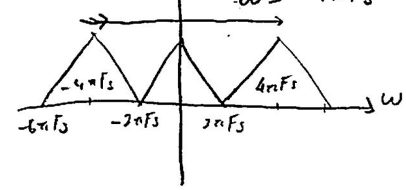
$$H_{eq}(s) = \frac{1}{2}\left(-\frac{1}{s-1} + \frac{s-2}{(s-2)^2+1} + \frac{1}{(s-2)^2+1}\right)$$

$$h_{eq}(t) = \frac{1}{2}u(t)\left(-e^{t} + e^{2t}\cos(t) + e^{2t}\sin(t)\right)$$

(b) (5 points) If F_s is the Nyquist rate of $x(t)$, determine in terms of F_s, the Nyquist rate of $x^3(t) * x^2(t)$.

$$X_2(j\omega) = \mathcal{F}\{x^2(t)\} = \frac{1}{2\pi}\left(X(j\omega) * X(j\omega)\right)$$

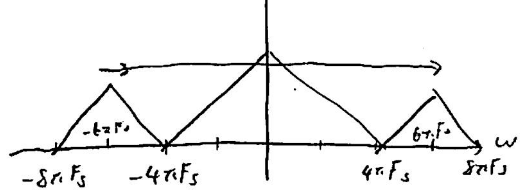
flip-and-drag
$\omega = -4\pi F_s$



$$X_2(j\omega) = \mathcal{F}\{x^2(t)\} = 0 \quad \forall \ |\omega| > 4\pi F_s.$$

$$X_3(j\omega) = \mathcal{F}\{x^3(t)\} = \frac{1}{2\pi}\left(X_2(j\omega) * X(j\omega)\right).$$

flip and drag



$$X_3(j\omega) = 0 \quad \forall \ |\omega| > 6\pi F_s.$$

$$x^3(t) * x^2(t) \Leftrightarrow \underbrace{X_2(j\omega)}_{|\omega| > 4\pi F_s} \underbrace{X_3(j\omega)}_{|\omega| > 6\pi F_s} = 0 \quad \forall \ |\omega| > 4\pi F_s$$

$\therefore$ The Nyquist rate is $\dfrac{4\pi F_s}{2\pi} = \boxed{2F_s}$