Total: 28 points

Signals and Systems

Midterm Exam

8:00 am - 10:00 am, November 1, 2016

Problem 1 (7 points) The following three questions are not related to each other.

1. (1.5 points) Consider the following signals: $x_1(t) = \text{sinc}(t)$, $x_2(t) = r(t) - 5 + r(-t)$, and $x_3(t) = te^{-3|t|}$. Which of these signals are even? which are odd?

2. (2.5 points) Determine whether the system

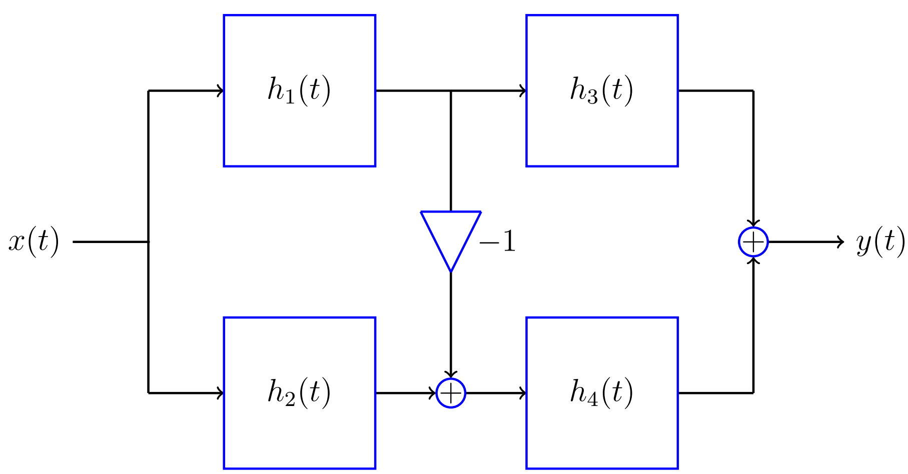
$$y(t) = \begin{cases} x(t-5) & \text{if} \quad |x(t)| \leq B \\ A|x(t)| & \text{otherwise} \end{cases},$$

where $|x(t)|$ is the magnitude of the input $x(t)$, is

   (a) Causal

   (b) Time invariant

3. (3 points) You are are told that the four blocks in the following block diagram represent LTI systems. Determine the expression for the impulse response of the overall system in terms of the impulse responses of the individual systems.



Solutions

1. Even signal: $x(-t) = x(t)$
   Odd signal: $x(-t) = -x(t)$

   $x_1(-t) = \text{sinc}(-t) = \frac{\sin(-t)}{-t} = \frac{-\sin(t)}{-t} = \frac{\sin(t)}{t} = \text{sinc}(t) = x_1(t)$
   $x_1(t)$ is even.
   $x_2(-t) = r(-t) - 5 + r(t) = r(t) - 5 + r(-t) = x_2(t)$
   $x_2(t)$ is even.
   $x_3(-t) = -te^{-3|-t|} = -te^{-3|t|} = -x_3(t)$
   $x_3(t)$ is odd.

   $x_1(t)$ and $x_2(t)$ are even while $x_3(t)$ is odd.

2. (a) For causality, we will check if $y(t)$ depends on future values of input signals. For computing $y(t)$, first we check if $|x(t)| \leq B$. This check does not require future values of the input signal. Once we have checked $|x(t)| \leq B$, $y(t)$ will take value either $x(t-5)$ or $A|x(t)|$, neither of them depends of future values of the input. Thus, our system is causal.

   (b) For time-invariance, we want response of $x(t-t_0)$ to be $y(t-t_0)$ if response of $x(t)$ is $y(t)$.

$$y(t) = \begin{cases} x(t-5) & \text{if} \quad |x(t)| \leq B \\ A|x(t)| & \text{otherwise} \end{cases}$$

   We calculate $y(t-t_0)$ from here as:

$$y(t-t_0) = \begin{cases} x(t-t_0-5) & \text{if} \quad |x(t-t_0)| \leq B \\ A|x(t-t_0)| & \text{otherwise} \end{cases}.$$

   Now we compute response of $x(t-t_0)$. Lets represent it by $\mathbb{S}\{x(t-t_0)\}$. To compute response of $x(t-t_0)$, we assume $x(t-t_0) = z(t)$, and compute response of $z(t)$ as:

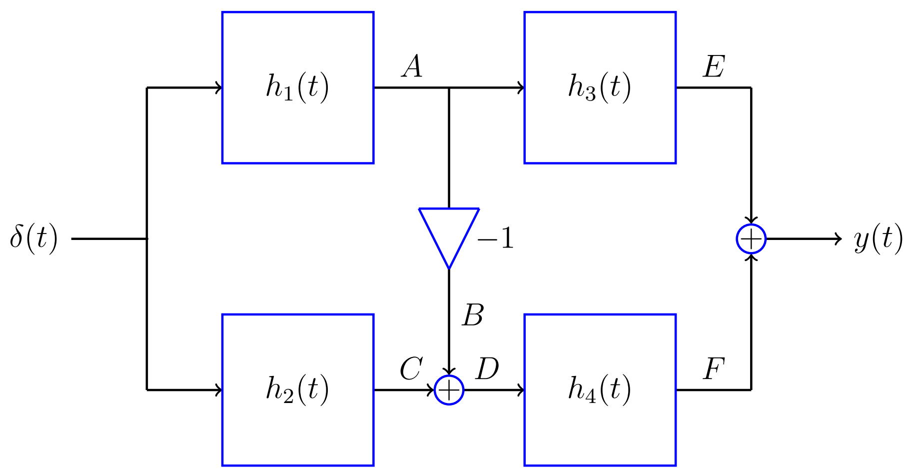
$$\mathbb{S}\{z(t)\} = \begin{cases} z(t-5) & \text{if} \quad |z(t)| \leq B \\ A|z(t)| & \text{otherwise} \end{cases}.$$

   We now substitute of $z(t) = x(t-t_0)$ to get

$$\mathbb{S}\{x(t-t_0)\} = \begin{cases} x(t-5-t_0) & \text{if} \quad |x(t-t_0)| \leq B \\ A|x(t-t_0)| & \text{otherwise} \end{cases}.$$

On comparing $\mathbb{S}\{x(t-t_0)\}$ and $y(t-t_0)$, we find that they are same. Thus, our system is time invariant.

3. For finding impulse response of the system, we will set input to $\delta(t)$ and compute the output. Lets represent outputs are various intermediate nodes by A, B, C, D, E and F as shown in the figure.

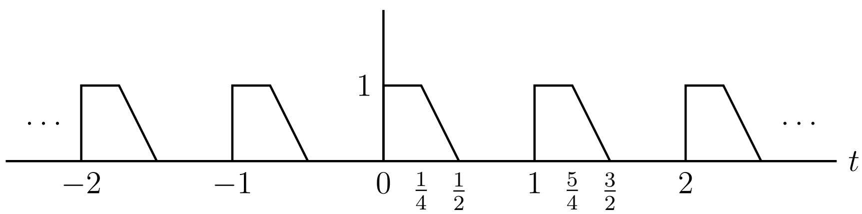


$$\begin{aligned}
A &= h_1(t) \\
B &= -A \\
&= -h_1(t) \\
C &= h_2(t) \\
D &= B + C \\
&= -h_1(t) + h_2(t) \\
E &= A * h_3(t) \\
&= h_1(t) * h_3(t) \\
F &= D * h_4(t) \\
&= (-h_1(t) + h_2(t)) * h_4(t)
\end{aligned}$$

Impulse response (response of δ_t) $h(t) =$ is $E + F$. Thus,

$$\begin{aligned}
h(t) &= E + F \\
&= h_1(t) * h_3(t) + (h_2(t) - h_1(t)) * h_4(t).
\end{aligned}$$

Problem 2 (7 points) (a) (4 points) Consider the signal in the following figure, that has period T=1. Calculate the Fourier Series coefficients $\{c_k\}$.



(b) (3 points) As we have discussed, the signal in the previous question also has as period all integer multiples of T, for instance, T=10 is also a period. What will happen if you calculate the Fourier Series coefficients, for the signal in part (a), assuming that T=10? Could you directly tell what these coefficients would be from the coefficients $\{c_k\}$ you calculated in part (a)?

Solutions (a) Let $x(t)$ be the signal in the given figure. $x(t)$ has period 1, so

$$\omega_0 = \frac{2\pi}{1} = 2\pi.$$

We compute c_0 and c_k, for $k \neq 0$ based on the following formulas.

$$c_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

Lets first calculate c_0.

$$\begin{aligned}
c_0 &= \frac{1}{T}\int_0^T x(t)dt = \frac{1}{1}\int_0^1 x(t)dt = \int_0^1 x(t)dt \\
&= \int_0^{0.25} 1dt + \int_{0.25}^{0.5}(-4t+2)dt \\
&= 0.25 + (-2t^2+2t)\Big|_{0.25}^{0.5} \\
&= 0.25 + [(-0.5+1) - (-0.125+0.5)] \\
&= 0.25 + [0.5 - (0.375)] \\
&= 0.25 + 0.125 = 0.375.
\end{aligned}$$

Now we calculate c_k for $k \neq 0$.

$$\begin{aligned}
c_k &= \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{1}\int_0^1 x(t)e^{-jk2\pi t}dt = \int_0^1 x(t)e^{-jk2\pi t}dt \\
&= \int_0^{0.25} 1e^{-jk2\pi t}dt + \int_{0.25}^{0.5} (-4t+2)e^{-jk2\pi t}dt \\
&= \left.\frac{e^{-jk2\pi t}}{-jk2\pi}\right|_0^{0.25} + (-4)\left.\left[t\frac{e^{-jk2\pi t}}{-jk2\pi} - \frac{e^{-jk2\pi t}}{(-jk2\pi)^2}\right]\right|_{0.25}^{0.5} + (2)\left.\left[\frac{e^{-jk2\pi t}}{-jk2\pi}\right]\right|_{0.25}^{0.5} \\
&= \frac{1}{jk2\pi}\left[1 - e^{-\frac{jk\pi}{2}}\right] + \frac{4}{jk2\pi}\left[\left(0.5e^{-jk\pi} + \frac{e^{-jk\pi}}{jk2\pi}\right) - \left(0.25e^{-\frac{jk\pi}{2}} + \frac{e^{-\frac{jk\pi}{2}}}{jk2\pi}\right)\right] \\
&\quad + \frac{2}{jk2\pi}\left[e^{-\frac{jk\pi}{2}} - e^{-jk\pi}\right] \\
&= \frac{1}{jk2\pi}\left[1 - \frac{4}{jk2\pi}e^{-\frac{jk\pi}{2}} + \frac{4}{jk2\pi}e^{-jk\pi}\right] \\
&= \frac{1}{jk2\pi}\left[1 - \frac{4}{jk2\pi}e^{-\frac{jk\pi}{2}} + \frac{4}{jk2\pi}e^{-jk\pi}\right]
\end{aligned}$$

(b) Now, our period is $T' = 10$. This will result in $\omega_0' = \frac{2\pi}{10} = \frac{\omega_0}{10}$. We assume that new coefficient with respect to ω_0' are d_k. Thus,

$$x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0' t} = \sum_{k=-\infty}^{\infty} d_k e^{\frac{jk\omega_0 t}{10}}.$$

From part (a), we also know that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

On comparing these two, we find that

$$d_k = \begin{cases} c_{\frac{k}{10}} & k = 10m \\ 0 & \text{otherwise} \end{cases}.$$

Problem 3 (7 points)

1. (2 points) Calculate the Fourier transform of the signal $\Pi(at-b)$, where $\Pi(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & \text{else} \end{cases}$ as a function of the parameters a and b.
2. (2 points) Let $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ be the derivative of the signal x(t), where $\Pi(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & \text{else} \end{cases}$. Calculate the Fourier transform of x(t).
3. (3 points) Calculate the following integral

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau$$

Solutions

1. From the Fourier transform pairs table, we see that Fourier transform of $\Pi(t)$ is $\text{sinc}(\frac{\omega}{2})$. Now we write following functions.

$$\begin{aligned} x_1(t) &= \Pi(t-b) \\ x_2(t) &= x_1(at) = \Pi(at-b) \end{aligned}$$

We use time shifting property of Fourier transform to find Fourier transform of $x_1(t)$ as $e^{-jb\omega}\,\text{sinc}(\frac{\omega}{2})$. Now we use scaling property of Fourier transform to find Fourier transform of $x_2(t)$ as $\frac{1}{|a|}e^{-\frac{jb\omega}{a}}\,\text{sinc}(\frac{\omega}{2a})$.

$$\begin{aligned} \Pi(t) &\Rightarrow \text{sinc}(\frac{\omega}{2}) \\ \Pi(t-b) &\Rightarrow e^{-jb\omega}\,\text{sinc}(\frac{\omega}{2}) \\ \Pi(at-b) &\Rightarrow \frac{1}{|a|}e^{-\frac{jb\omega}{a}}\,\text{sinc}(\frac{\omega}{2a}) \end{aligned}$$

Thus, Fourier transform of $\Pi(at-b)$ is $\frac{1}{|a|}e^{-\frac{jb\omega}{a}}\,\text{sinc}(\frac{\omega}{2a})$.

2. We use results from part(1) to find Fourier transforms of $\Pi(t-2)$ and $\Pi(\frac{t-5}{2})$.

$$\begin{aligned} \Pi(t-2) &\Rightarrow e^{-j2\omega}\,\text{sinc}(\frac{\omega}{2}) \\ \Pi(\frac{t-5}{2}) &\Rightarrow 2e^{-j5\omega}\,\text{sinc}(\omega). \end{aligned}$$

Now we use linearly property of Fourier tranform to prove that Fourier transform of $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ is :

$$\Pi(t-2) - 0.5\Pi(\frac{t-5}{2}) \Rightarrow e^{-j2\omega}\,\text{sinc}(\frac{\omega}{2}) - e^{-j5\omega}\,\text{sinc}(\omega).$$

Since $y(t)$ is derivative of $x(t)$, by using differentiation property, Fourier transform of $x(t)$ is

$$\frac{1}{j\omega}\left(e^{-j2\omega}\operatorname{sinc}(\frac{\omega}{2}) - e^{-j5\omega}\operatorname{sinc}(\omega)\right).$$

3. We first simplify the following integral.

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau \\
&= \int_{-\infty}^{\infty} \frac{\operatorname{sinc}(18\tau)}{\tau} \frac{\operatorname{sinc}(4\tau)}{\tau} e^{i11\tau} d\tau \\
&= \int_{-\infty}^{\infty} (18\operatorname{sinc}(18\tau))\,(4\operatorname{sinc}(4\tau))\, e^{i11\tau} d\tau \\
&= 2\pi\ \frac{1}{2\pi}\int_{-\infty}^{\infty} (18\operatorname{sinc}(18\omega))\,(4\operatorname{sinc}(4\omega))\, e^{i11\omega} d\omega \\
&= 2\pi\ f(11),
\end{aligned}$$

where $f(11)$ is the inverse Fourier transform of $(18\operatorname{sinc}(18\omega))\,(4\operatorname{sinc}(4\omega))$ at $t = 11$. From the convolution property of Fourier transform $f(t)$ is $p(t) * q(t)$, where $p(t)$ is inverse Fourier transform of $(18\operatorname{sinc}(18\omega))$, and $q(t)$ is inverse Fourier transform of $(4\operatorname{sinc}(4\omega))$.

From the Fourier transform pair table, we get

$$\begin{aligned}
p(t) &= 0.5\ \Pi\left(\frac{t}{36}\right) \\
q(t) &= 0.5\ \Pi\left(\frac{t}{8}\right).
\end{aligned}$$

Now we calculate $f(11)$ that is convolution of $p(t)$ and $q(t)$ evaluated at 11.

$$\begin{aligned}
f(11) &= 0.25 \int\limits_{\tau=-\infty}^{\infty} \Pi\left(\frac{\tau}{36}\right)\ \Pi\left(\frac{11-\tau}{8}\right) d\tau \\
&= 0.25 \int\limits_{\tau=7}^{15} \Pi\left(\frac{\tau}{36}\right)\ \Pi\left(\frac{11-\tau}{8}\right) d\tau \\
&= 0.25 \int\limits_{\tau=7}^{15} 1.\ 1\ d\tau \\
&= 2.
\end{aligned}$$

Thus

$$y(t) = 2\pi f(11) = 4\pi.$$

Problem 4 (7 points) Sometimes we work with systems that take as input two signals, say $f(t)$ and $g(t)$ and produce at their output one signal, say $y(t)$. One way of analyzing such systems is by assuming they take as input a 2×1 vector $x(t)$ that has as elements the signals $f(t)$ and $g(t)$; we then apply the definitions for linearity and time invariance on the vector input $x(t)$.

Consider a system that takes as input two real signals $f(t)$ and $g(t)$ and calculates as output their inner product $y(t)$ defined as

$$y(t) = (f, g) = \int_{-\infty}^{\infty} f(t)g(t)\, dt$$

Recall that the time reverse of a signal $x(t)$ is the signal $x(-t)$, and the time shifted version of $x(t)$ by some constant t_0 is the signal $x(t - t_0)$.

(a) If both $f(t)$ and $g(t)$ are time reversed, what happens to their inner product?

(b) Assume that only one of $f(t)$ and $g(t)$ is time reversed, does the outcome depend on which one was reversed or no?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) Assume that you can use as blocks the following systems: a block that takes as input a signal and time reverses it, a block that takes as input 2 signals $x_1(t)$ and $x_2(t)$ and outputs the signal $y(t) = x_1(t) * x_2(t)$ that is their convolution, a block that takes as input a signal and delays it by a fixed amount t_0 we can select, and a block that takes as input a signal $x(t)$ and outputs the constant value $\int_{-\infty}^{\infty} x(t)\delta(t - t_1)dt$ for a constant t_1 we can select. Can you connect (some of) these blocks to create a system that takes as input two signals and outputs their inner product value?

(e) Is the system that implements the inner product time invariant? Is it linear?

Solutions (a) Both $f(t)$ and $g(t)$ are time reversed. This means that new $f'(t)$ and $g'(t)$ are

$$f'(t) = f(-t), \quad g'(t) = g(-t).$$

New inner product (f', g') is:

$$\begin{aligned}
(f', g') &= \int_{-\infty}^{\infty} f'(t)g'(t)\, dt \\
&= \int_{-\infty}^{\infty} f(-t)g(-t)\, dt.
\end{aligned}$$

Now, we substitute $t' = -t$, and we get:

$$\begin{aligned}
(f', g') &= \int_{-\infty}^{\infty} f(-t)g(-t)\,dt \\
&= -\int_{\infty}^{-\infty} f(t')g(t')\,dt' \\
&= \int_{-\infty}^{\infty} f(t')g(t')\,dt' \\
&= \int_{-\infty}^{\infty} f(t)g(t)\,dt \\
&= (f, g).
\end{aligned}$$

Thus, by time reversing both f and g, inner product remains unchanged.

(b) If f is time reversed, outcome will be

$$(f', g) = \int_{-\infty}^{\infty} f(-t)g(t)\,dt.$$

On the other hand, if g is time reversed, outcome will be

$$(f, g') = \int_{-\infty}^{\infty} f(t)g(-t)\,dt.$$

Now we substitute $t' = -t$ in above equation to get,

$$\begin{aligned}
(f, g') &= -\int_{\infty}^{-\infty} f(-t')g(t')\,dt' \\
&= \int_{-\infty}^{\infty} f(-t')g(t')\,dt' \\
&= \int_{-\infty}^{\infty} f(-t)g(t)\,dt \\
&= (f', g).
\end{aligned}$$

Thus, it does not matter which one time reversed, outcome in both cases are same.

(c) Let $f'(t) = f(t - t_0)$ and $g'(t) = g(t - t_0)$. The new inner product will be,

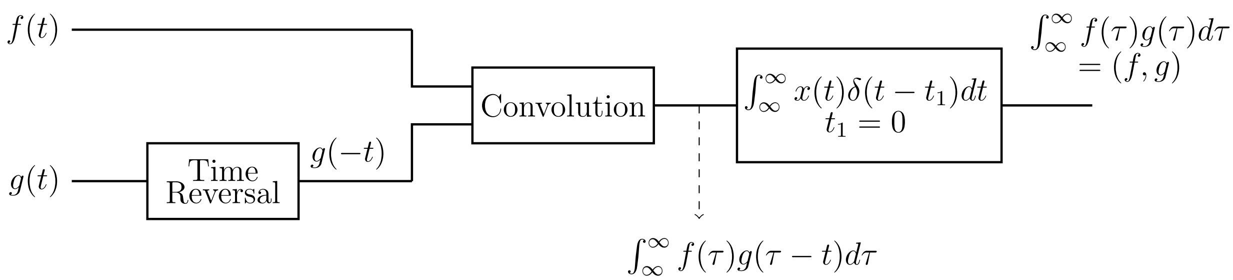
$$(f', g') = \int_{-\infty}^{\infty} f(t - t_0)g(t - t_0)\,dt.$$

Now we substitute $t' = t - t_0$ in above equation to get,

$$\begin{aligned}(f, g') &= \int_{-\infty}^{\infty} f(t')g(t')\,dt' \\ &= \int_{-\infty}^{\infty} f(t)g(t)\,dt \\ &= (f, g).\end{aligned}$$

Thus, when time shifting both f and g by same amount, the inner product remains unchanged.

(d)



(e) Time Invariance: For input $x(t) = [f(t)\ g(t)]$, we get a constant output $y(t) = (f, g)$. If we change input to $x(t - t_0)$, we know from part(c) that output still remains the same, that is $\mathbb{S}\{x(t - t_0)\} = y(t)$. Since output is constant, $y(t) = y(t - t_0)$. Thus, our system is time-invariant.
Linearity:

$$\begin{aligned}x_1(t) &= [f_1(t)\ g_1(t)] \ \Rightarrow\ (f_1, g_1) \\ x_2(t) &= [f_2(t)\ g_2(t)] \ \Rightarrow\ (f_2, g_2) \\ \alpha x_1(t) + \beta x_2(t) &= [\alpha f_1(t) + \beta f_2(t) \ \ \alpha g_1(t) + \beta g_2(t)] \\ &\Rightarrow (\alpha f_1 + \beta f_2, \alpha g_1 + \beta g_2) \\ &= \alpha^2 (f_1, g_1) + \beta^2 (f_2, g_2) + \alpha\beta(f_1, g_2) + \alpha\beta(f_2, g_1) \\ &\neq \alpha(f_1, g_1) + \beta(f_2, g_2) \\ &= \alpha y_1(t) + \beta y_2(t).\end{aligned}$$

Thus our system is not linear.

1 Properties of Fourier Series

$x(t)$ and $y(t)$ are periodic signals of period T. ($\omega_0 = \frac{2\pi}{T}$)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\, dt$$

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}, \quad d_k = \frac{1}{T}\int_T y(t)e^{-jk\omega_0 t}\, dt$$

Property	Signal	k^{th} Fourier coefficient
	$x(t)$ $y(t)$	c_k d_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha c_k + \beta d_k$
Time-Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} c_k$
Conjugation	$x^*(t)$	c^*_{-k}
Time-Reversal	$x(-t)$	c_{-k}
Time-Scaling	$x(\alpha t), \alpha > 0$ Period : $\frac{T}{\alpha}$	c_k
Conjugate-Symmetry	$x(t)$ is real	$c_k = c^*_{-k}$
Even-Odd Signals	$x(t)$ is real and even $x(t)$ is real and odd	c_k is real and even c_k is purely imaginary and odd

Parsevals Relation for Periodic Signals: $\frac{1}{T}\int\limits_T |x(t)|^2 dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

2 Fourier Transform Formulas

Fourier transform formulas (using ω):

Synthesis equation (Inverse Fourier Transform): $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Analysis equation (Fourier Transform): $X(\omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Fourier transform formulas (using f):

Synthesis equation (Inverse Fourier Transform): $x(t) = \int\limits_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$

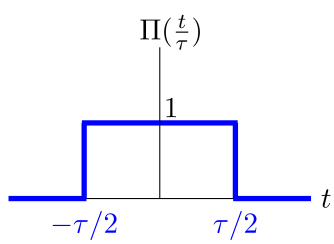
Analysis equation (Fourier Transform): $X(f) = \int\limits_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$

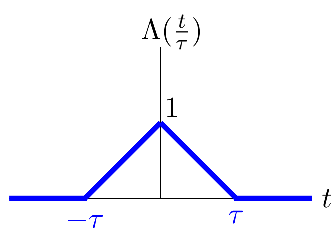
3 Fourier Transform Properties

Property	Signal	Fourier Transform
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$
Conjugate symmetry	$x(t)$ is real	$X^*(\omega) = X(-\omega)$
Conjugate anti-symmetry	$x(t)$ is purely imaginary	$X^*(\omega) = -X(-\omega)$
Even and real signal	$x(-t) = x(t)$	$\text{Im}\{X(\omega)\} = 0$
Odd and real signal	$x(-t) = -x(t)$	$\text{Re}\{X(\omega)\} = 0$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Modulation Property	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$
Time and frequency scaling	$x(at)$	$\frac{1}{\|a\|}X(\frac{\omega}{a})$
Differentiation in time	$\frac{d^n}{dt^n}[x(t)]$	$(j\omega)^n X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$

$$\text{Parseval's theorem: } \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

4 Fourier Transform pairs





$$\textbf{We define, sinc}(x) := \frac{\sin(x)}{x}$$

Name	Signal	Fourier Transform
Rectangular pulse	$x(t) = A\ \Pi(t/\tau)$	$X(\omega) = A\tau\ \text{sinc}(\frac{\omega\tau}{2})$
Triangular pulse	$x(t) = A\ \Lambda(t/\tau)$	$X(\omega) = A\tau\ \text{sinc}^2(\frac{\omega\tau}{2})$
Right-sided exponential	$x(t) = e^{-at}u(t)$	$X(\omega) = \frac{1}{a+j\omega}$
Two-sided exponential	$x(t) = e^{-a\|t\|}$	$X(\omega) = \frac{2a}{a^2+\omega^2}$
Unit impulse	$x(t) = \delta(t)$	$X(\omega) = 1$
Sinc function	$x(t) = \text{sinc}(\pi t)$	$X(\omega) = \Pi(\frac{\omega}{2\pi})$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(\omega) = 2\pi\delta(\omega)$
Unit-step function	$x(t) = u(t)$	$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$