Total: 28 points

Signals and Systems

Midterm Exam

8:00 am - 10:00 am, November 1, 2016

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This exam has 4 problems, for a total of 28 points.

Closed book. No calculators. No electronic devices.

One page, letter-size, one-side cheat-sheet allowed.

Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.

The Fourier series and transform tables are provided in the last two pages.

Please, write your name and UID on the top of each loose sheet!

GOOD LUCK!

Problem	Points	Total Points
1	7	7
2	7	7
3	4	7
4	7	7
Total	25	28

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Problem 1 (7 points) The following three questions are not related to each other.

1. (1.5 points) Consider the following signals: $x_1(t) = \text{sinc}(t)$, $x_2(t) = r(t) - 5 + r(-t)$, and $x_3(t) = te^{-3|t|}$. Which of these signals are even? which are odd?

2. (2.5 points) Determine whether the system

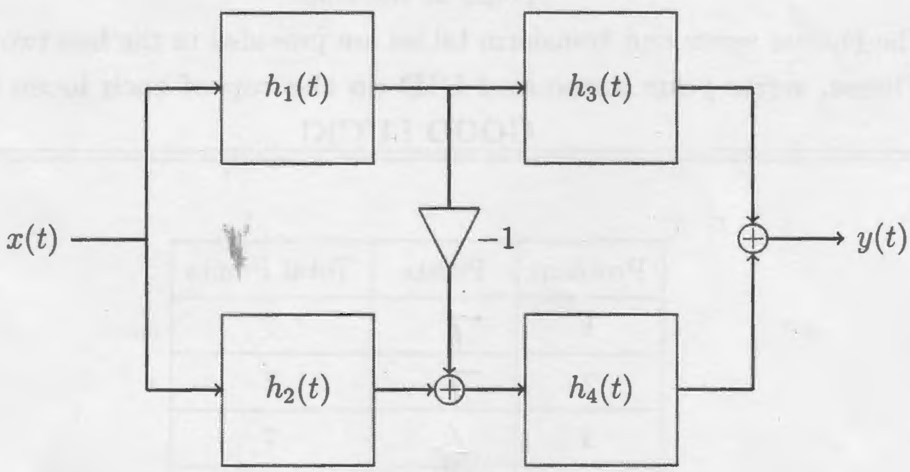
$$y(t) = \begin{cases} x(t-5) & \text{if } |x(t)| \leq B \\ A|x(t)| & \text{otherwise} \end{cases},$$

where $|x(t)|$ is the magnitude of the input $x(t)$, is

   (a) Causal

   (b) Time invariant

3. (3 points) You are are told that the four blocks in the following block diagram represent LTI systems. Determine the expression for the impulse response of the overall system in terms of the impulse responses of the individual systems.



1. ① $\mathrm{sinc}(-t) = \frac{\sin(-t)}{-t} = \frac{-\sin t}{-t} = \frac{\sin t}{t} = \mathrm{sinc}(t)$

$x_1(t)$ is even

$x_2(-t) = r(-t) - 5 + r(t) = r(t) - 5 + r(-t) = x_2(t)$.

$x_2(t)$ is even

$x_3(-t) = -te^{-3|-t|} = -te^{-3|t|} = -x_3(t)$

$x_3(t)$ is odd

So $x_1(t)$ $x_2(t)$ are even, $x_3(t)$ is odd

1.5

② (a) $y(t)$ is causal because it only depends on $t-5$ and t

(b) let $z(t) = x(t-t_0)$

$$y(t) = \begin{cases} z(t-5) & \text{if } |z(t)| \le B \\ A|z(t)| & \text{otherwise} \end{cases}$$

$$= \begin{cases} x(t-t_0-5) & \text{if } |x(t-t_0)| \le B \\ A|x(t-t_0)| & \text{otherwise} \end{cases}$$

$$y(t-t_0) = \begin{cases} x(t-t_0-5) & \text{if } |x(t-t_0)| \le B \\ A|x(t-t_0)| & \text{otherwise} \end{cases}$$

output are same, so it's time invariant

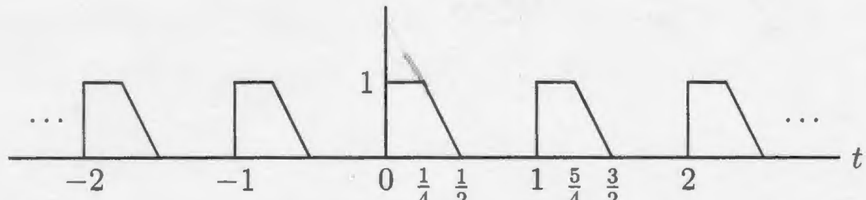
2.5

③

$y(t) = h_1(t) * h_3(t) + [h_2(t) - h_1(t)] * h_4(t)$

3

Problem 2 (7 points) (a) (4 points) Consider the signal in the following figure, that has period T=1. Calculate the Fourier Series coefficients $\{c_k\}$.



$\Omega_1 = \frac{2\pi}{T}$

$\sum_k X_k e^{jk\Omega_1 t}$

(b) (3 points) As we have discussed, the signal in the previous question also has as period all integer multiples of T, for instance, T=10 is also a period. What will happen if you calculate the Fourier Series coefficients, for the signal in part (a), assuming that T=10? Could you directly tell what these coefficients would be from the coefficients $\{c_k\}$ you calculated in part (a)?

(a) $C_k = \frac{1}{T}\int_0^T x(t) e^{-i\frac{2\pi k t}{T_0}} dt$. $\begin{cases} 1 = \frac{1}{4}k + b \\ 0 = \frac{1}{2}k + b \end{cases}$ $\begin{cases} k = -4 \\ b = 2 \end{cases}$

$= \int_0^1 x(t) e^{-i2\pi kt} dt$ $\quad y = -4x+2$.

$= \int_0^{\frac{1}{4}} e^{-i2\pi kt} dt + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4t+2) e^{-i2\pi kt} dt \quad -4\int_{\frac{1}{4}}^{\frac{1}{2}} t e^{-i2\pi kt}$

$= \frac{e^{-i2\pi kt}}{-i2\pi k}\Big|_0^{\frac{1}{4}} + -4\left(\frac{t}{-i2\pi k} e^{-i2\pi kt} + \frac{1}{4\pi^2k^2} e^{-i2\pi kt}\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}} + 2\frac{e^{-i2\pi kt}}{-i2\pi k}\Big|_{\frac{1}{4}}^{\frac{1}{2}}$

$= \frac{e^{-i\frac{1}{2}\pi k} - 1}{-i2\pi k} + 4\left(\frac{e^{-i\pi k}}{-i4\pi k} + \frac{1}{4\pi^2k^2} e^{-i\pi k} - \frac{1}{-i8\pi k} e^{-i\frac{1}{2}\pi k} - \frac{1}{4\pi^2k^2} e^{-i\frac{1}{2}\pi k}\right)$

$+ 2\left(\frac{e^{-i\pi k}}{-i2\pi k} - \frac{e^{-i\frac{1}{2}\pi k}}{-i2\pi k}\right)$

4

$= \frac{e^{-i\frac{1}{2}\pi k} - 1}{-i2\pi k} + \frac{e^{-i\pi k}}{i\pi k} - \frac{1}{\pi^2k^2} e^{-i\pi k} - \frac{e^{-i\frac{1}{2}\pi k}}{i2\pi k} + \frac{e^{-i\frac{1}{2}\pi k}}{\pi^2k^2} - \frac{e^{-i\pi k}}{i\pi k} + \frac{e^{-i\frac{1}{2}\pi k}}{i\pi k}$

$= \frac{1 - 2e^{-i\frac{1}{2}\pi k}}{i2\pi k} + \frac{e^{-i\frac{1}{2}\pi k}}{i\pi k} + \frac{e^{-i\frac{1}{2}\pi k} - e^{-i\pi k}}{\pi^2k^2}$

$= \frac{1}{i2\pi k} + \frac{e^{-i\frac{1}{2}\pi k} - e^{-i\pi k}}{\pi^2k^2}$

$C_0 = -2t^2 + 2t\Big|_{\frac{1}{4}}^{\frac{1}{2}} = -\frac{1}{2} + 1 - \left(-\frac{1}{8} + \frac{1}{2}\right)$

$= -\frac{1}{2} + 1 + \frac{1}{8} - \frac{1}{2}$

$= \frac{1}{8}$

$= \sum_k C_{k/10} e^{j\frac{k}{10}\Omega_1 t}$.

(b) . $X(t) = \sum_k C_k e^{jk\Omega_1 t}$ $\quad \Omega_1 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi \quad \frac{2\pi}{10} = \frac{1}{10}\Omega_1$

so $X_k = C_{k/10}$ for $k = 10s$ s is integer.

3

Problem 3 (7 points)

1. (2 points) Calculate the Fourier transform of the signal $\Pi(at-b)$, where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$ as a function of the parameters a and b.

2. (2 points) Let $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ be the derivative of the signal x(t), where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$. Calculate the Fourier transform of x(t).

3. (3 points) Calculate the following integral

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau$$

3. ① $\Pi(\omega) = \int_{-0.5}^{0.5} e^{-j\omega t} dt$

$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-0.5}^{0.5}$

$= -\frac{1}{j\omega} e^{-\frac{1}{2}j\omega} + \frac{1}{j\omega} e^{\frac{1}{2}j\omega}$

$= \frac{1}{j\omega} (e^{\frac{1}{2}j\omega} - e^{-\frac{1}{2}j\omega})$

$= \frac{1}{j\omega} \sin(\frac{1}{2}\omega) \cdot 2j$

$= \frac{2}{\omega} \sin(\frac{1}{2}\omega)$

$\Pi(t-b) \longleftrightarrow \frac{2}{\omega} \sin(\frac{1}{2}\omega) \cdot e^{-j\omega b}$

$\Pi(at-b) \longleftrightarrow \frac{1}{|a|} \frac{2a}{\omega} \sin(\frac{\omega}{2a}) e^{-j\frac{\omega}{a}b}$

2

② $y(\omega) = \frac{2}{\omega}\sin(\frac{1}{2}\omega)e^{-j\omega 2} - \sin(\omega)e^{-j\omega 10}$

$x(\omega) = \frac{y(\omega)}{i\omega} = \frac{2}{i\omega^2}\sin(\frac{1}{2}\omega)e^{-2j\omega} - \frac{e^{-10j\omega}}{i\omega}\sin(\omega)$

1.5

③ $\cancel{\int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau = \int_{-\infty}^{\infty} \frac{(e^{i18\tau} - e^{-i18\tau})(e^{i4\tau} - e^{-i4\tau})}{-4\tau^2} e^{i11\tau}}$

$\cancel{= \int_{-\infty}^{\infty} \frac{e^{-72\tau^2} - e^{72\tau^2} - e^{72\tau^2} + e^{-72\tau^2}}{-4\tau^2} e^{i11\tau}}$

$\cancel{= \int_{-\infty}^{\infty} \frac{e^{-72\tau^2} - e^{72\tau^2}}{-2\tau^2} e^{i11\tau} = (e^{i36\tau} + e^{-i36\tau})(e^{i36\tau} - e^{-i36\tau})}$

$f(t)$
$g(t)$

Problem 4 (7 points) Sometimes we work with systems that take as input two signals, say $f(t)$ and $g(t)$ and produce at their output one signal, say $y(t)$. One way of analyzing such systems is by assuming they take as input a 2×1 vector $x(t)$ that has as elements the signals $f(t)$ and $g(t)$; we then apply the definitions for linearity and time invariance on the vector input $x(t)$.

Consider a system that takes as input two real signals $f(t)$ and $g(t)$ and calculates as output their inner product $y(t)$ defined as

$$y(t) = (f, g) = \int_{-\infty}^{\infty} f(t)g(t)\,dt$$

Recall that the time reverse of a signal $x(t)$ is the signal $x(-t)$, and the time shifted version of $x(t)$ by some constant t_0 is the signal $x(t - t_0)$.

(a) If both $f(t)$ and $g(t)$ are time reversed, what happens to their inner product?

(b) Assume that only one of $f(t)$ and $g(t)$ is time reversed, does the outcome depend on which one was reversed or no?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

$x(-t) = -x(t)$

(d) Assume that you can use as blocks the following systems: a block that takes as input a signal and time reverses it, a block that takes as input 2 signals $x_1(t)$ and $x_2(t)$ and outputs the signal $y(t) = x_1(t) * x_2(t)$ that is their convolution, a block that takes as input a signal and delays it by a fixed amount t_0 we can select, and a block that takes as input a signal $x(t)$ and outputs the constant value $\int_{-\infty}^{\infty} x(t)\delta(t - t_1)dt$ for a constant t_1 we can select. Can you connect (some of) these blocks to create a system that takes as input two signals and outputs their inner product value?

$x(t_1)$

(e) Is the system that implements the inner product time invariant? Is it linear?

$\int x_1(\tau)\, x_2(t-\tau)\, d\tau$

7/7

(a) $\int_{-\infty}^{\infty} f(-t)\,g(-t)\,dt$

let $-t = t'$ $\quad dt' = -dt$

$= -\int_{\infty}^{-\infty} f(t')\,g(t')\,dt'$

$= \int_{-\infty}^{\infty} f(t')\,g(t')\,dt'$

The inner output will not change.

(b) $\int_{-\infty}^{\infty} f(-t)\,g(t)\,dt = \int_{-\infty}^{\infty} f(t')\,g(-t')\,dt'$

No, the output doesn't depend on which one has changed.

(c). $\int_{-\infty}^{\infty} f(t-t_0)\,g(t-t_0)\,dt$

let $t' = t - t_0$

$dt' = dt$

$= \int_{-\infty}^{\infty} f(t')\,g(t')\,dt'$

The inner product will not change, it will change if $t_0 \to \infty$

(d). $y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x(\tau)\,x_2(t-\tau)\,d\tau$.

X_1

x

$\int_{-\infty}^{\infty} X_1(t)\,X_2(t)\,dt$

(e). from (c) we know that $\int_{-\infty}^{\infty} f(t-t_0)\,g(t-t_0)\,dt$

$= \int_{-\infty}^{\infty} f(t-t_0)\,g(t-t_0)\,dt.$

So it's time-invariant

if $a = f_1(t) + f_2(t)$ $\qquad \int a f_1(t) \cdot b\,g(t)\,dt = \int$.

$b = g_1(t) + g_2(t)$

$\int_{-\infty}^{\infty} f_1(t)\,g_1(t)\,dt + \int_{-\infty}^{\infty} f_2(t)\,g_2(t)\,dt \neq \int_{-\infty}^{\infty} (f_1(t)+f_2(t))(g_1(t)+g_2(t))\,dt$

So it's not linear.