Total: 28 points

Signals and Systems

Midterm Exam

Solutions

Problem 1 (12 points)

I. (3 *points*) Evaluate the following convolutions:

- $u(t) \star e^{-t}u(t)$,
- $u(t) \star (u(t-1) - u(t-2))$.

II. (9 *points*) Consider the following three LTI systems:

- $\mathcal{S}_1$: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- $\mathcal{S}_2$: LTI system with impulse response $h_2(t) = e^{-t}u(t)$.
- $\mathcal{S}_3$: LTI system with impulse response $h_3(t) = u(-2t+2) - u(-2t+4)$.

The three systems are now interconnected as shown in Figure 1. What is the impulse response $h(t)$ of the overall LTI system (i.e. from $x(t)$ to $z(t)$)? Is the overall system stable? Causal?

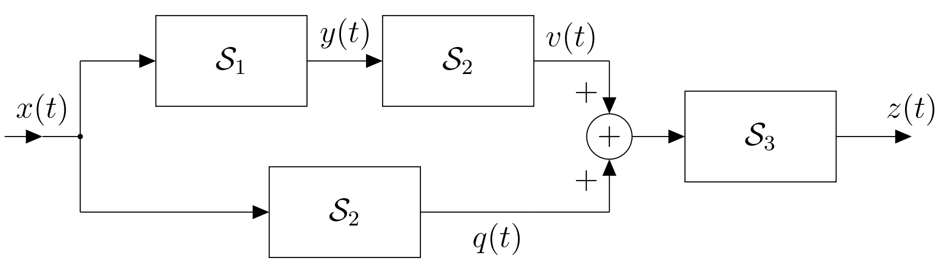


Figure 1: The LTI system in Problem 1

Solution:

I. The convolutions give the following signals:

- $u(t) \star e^{-t}u(t) = \left(\int_0^t e^{-\tau}\right) u(t) = (1 - e^{-t})u(t)$,
- $u(t) \star (u(t-1) - u(t-2)) = (t-1)u(t-1) - (t-2)u(t-2)$.

II. The first system has the following impulse response: $h_1(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$. Moreover the impulse response of the third system can be equivalently written as: $h_3(t) = u(-2t+2) - u(-2t+4) = u(-t+1) - u(-t+2) = -(u(t-1) - u(t-2))$. Thus the overall impulse response is:

$$
\begin{aligned}
&[h_1(t) \star h_2(t) + h_2(t)] \star h_3(t) \\
&= \left[u(t) \star (e^{-t}u(t)) + e^{-t}u(t)\right] \star (-(u(t-1) - u(t-2))) \\
&= -(t-1)u(t-1) + (t-2)u(t-2)
\end{aligned}
$$

The overall system is causal since $h(t) = 0$ for all $t < 0$. The overall system is not stable since $\int |h(t)|dt = \int_1^2 (t-1)dt + \int_2^\infty 1dt$ which does not converge, thus the system is unstable.

Problem 2 (8 points)

Figure 2 shows three real signals. All the signals have finite time support, i.e. the signals are zero at any time not shown in the figure.

(a) Determine which, if any, of the real signals depicted in Figure 2 have Fourier transforms that satisfy each of the following conditions (and very briefly say why):

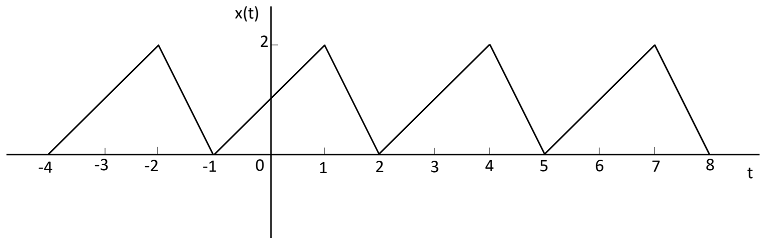
1. (1.5 *points*) $\mathcal{R}e\{X(\omega)\} \neq 0$.
2. (1.5 *points*) $\mathcal{I}m\{X(\omega)\} \neq 0$.
3. (1.5 *points*) There exists a real number such that $e^{j\alpha\omega}X(\omega)$ is real.
4. (1.5 *points*) $\int_{-\infty}^{\infty} X(\omega)d\omega = 0$.
5. (1.5 *points*) $\int_{-\infty}^{\infty} \omega X(\omega)d\omega = 0$.

(b) (1 *points*) Calculate, for the signal in Figure 2 (b) the value of $\int_{-\infty}^{\infty} \cos(2\omega)X(\omega)d\omega$. (Hint: you do not need to calculate $X(\omega)$).

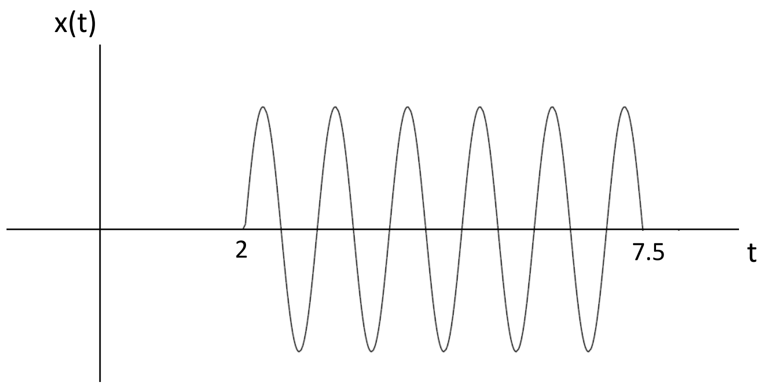
Solution:

(a) Since $x(t)$ is real, $\mathcal{R}e\{X(\omega)\}$ is the Fourier transform of the even component of $x(t)$ and that $j\mathcal{I}m\{X(\omega)\}$ is the Fourier transform of the odd component of $x(t)$.

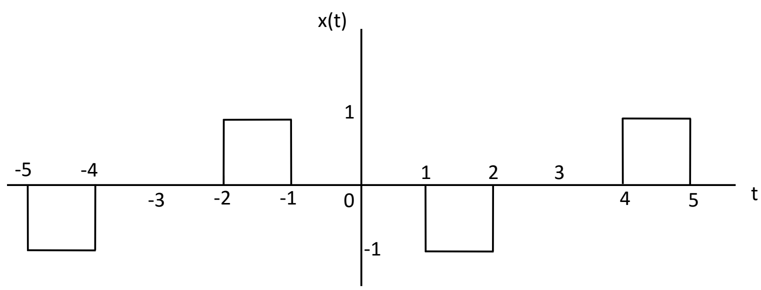
1. $\mathcal{R}e\{X(\omega)\} \neq 0$: this means that $x(t)$ has an even component, thus $x(t)$ cannot be odd. Therefore the possible signals are (a) and (b).
2. $\mathcal{I}m\{X(\omega)\} \neq 0$: this means that $x(t)$ has an odd component, thus $x(t)$ cannot be even. Therefore the possible signals are (a), (b) and (c).
3. For $e^{j\alpha\omega}X(\omega)$ to be real, its inverse Fourier transform should be even, i.e., $x(t+a)$ should be even. For the signals in (a) and (c), any shift will not make this signal even, and therefore there is no a for these signals such that $x(t+a)$ is even. For the signal in (b), a left shift by 4.75 makes the signal even, thus a possible value for a is 4.75.



(a)



(b)



(c)

Figure 2: Problem 2

4. Since

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

we can conclude that

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$$

Therefore, if $\int_{-\infty}^{\infty} X(\omega)d\omega = 0$, then $x(0) = 0$. Signals in (b) and (c) satisfy this condition.

5. Let $y(t) = \frac{dx(t)}{dt}$, then $Y(\omega) = j\omega X(\omega)$. Thus,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega X(\omega)e^{j\omega t}d\omega$$

Moreover,

$$y(0) = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega X(\omega)d\omega$$

Therefore, if $\int_{-\infty}^{\infty} \omega X(\omega)d\omega = 0$, then $y(0) = 0$, or $\frac{dx(t)}{dt}|_{t=0} = 0$. The signals in (c) and (b) satisfy this condition.

(b) $\int_{-\infty}^{\infty} \cos(2\omega)X(\omega)d\omega = \int_{-\infty}^{\infty} \frac{e^{j2\omega}+e^{-j2\omega}}{2}X(\omega)d\omega = \pi x(2) + \pi x(-2) = 0.$