Total: 28 points

Signals and Systems

Midterm Exam Solutions

October 30, 2013

Problem 1 (7 points)

I.

1. Let $x_1(t) = \frac{B}{2}$ and $x_2(t) = \frac{3B}{4}$. Then $y_1(t) = T[x_1(t)] = \frac{B}{2}$ and $y_2(t) = T[x_2(t)] = \frac{3B}{4}$. Since $T[x_1(t) + x_2(t)] = B$ but $y_1(t) + y_2(t) = \frac{5B}{4} \neq B$, $T[x_1(t) + x_2(t)] \neq y_1(t) + y_2(t)$ and the system is therefore $\boxed{\textit{nonlinear}}$.

2. It is obvious from the definition of $y(t)$ that the current output of the system depends only on the current input. The system is therefore $\boxed{\textit{causal}}$.

3. From the definition of $y(t)$ we know that regardless of the input, the output is bounded between $-B$ and B ($|y(t)| \leq B$). Therefore, the system is $\boxed{\textit{stable}}$.

4.

$$T[x(t-\tau)] = \begin{cases} x(t-\tau) & \text{if } |x(t-\tau)| \leq B \\ \text{sign}(x(t-\tau)) & \text{otherwise} \end{cases}$$

$$y(t-\tau) = \begin{cases} x(t-\tau) & \text{if } |x(t-\tau)| \leq B \\ \text{sign}(x(t-\tau)) & \text{otherwise} \end{cases}$$

Since $T[x(t-\tau)] = y(t-\tau)$, the system is $\boxed{\textit{time invariant}}$.

II.

1.

$$y(-t) = \begin{cases} 2B\cos(-2\pi t) & \text{if } |2B\cos(-2\pi t)| \leq B \\ \text{sign}(2B\cos(-2\pi t)) & \text{otherwise} \end{cases}$$

Since $\cos(-2\pi t) = \cos(2\pi t)$,

$$y(-t) = \begin{cases} 2B\cos(2\pi t) & \text{if } |2B\cos(2\pi t)| \le B \\ \text{sign}(2B\cos(2\pi t)) & \text{otherwise} \end{cases} = y(t)$$

Therefore, the resulting signal is $\boxed{even}$.

2.

$$y(t+1) = \begin{cases} 2B\cos(2\pi(t+1)) & \text{if } |2B\cos(2\pi(t+1)| \le B \\ \text{sign}(2B\cos(2\pi(t+1))) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2B\cos(2\pi t + 2\pi)) & \text{if } |2B\cos(2\pi t + 2\pi)| \le B \\ \text{sign}(2B\cos(2\pi t + 2\pi)) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2B\cos(2\pi t)) & \text{if } |2B\cos(2\pi t)| \le B \\ \text{sign}(2B\cos(2\pi t) & \text{otherwise} \end{cases} = y(t)$$

Therefore, the resulting signal is $\boxed{periodic}$.

3. Since $y(t)$ is periodic, it is not zero when $t < 0$. Therefore, the resulting signal is $\boxed{noncasual}$.

Problem 2 (8 points)

I.

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}\,dt$$

Let $\tau = at$. Then $t = \frac{\tau}{a}$ and $dt = \frac{1}{a}d\tau$. so

$$\begin{aligned} \int_0^\infty f(at)e^{-st}\,dt &= \int_0^\infty f(\tau)e^{-s(\frac{\tau}{a})}\,\frac{1}{a}d\tau \\ &= \frac{1}{a}\int_0^\infty f(\tau)e^{-\tau(\frac{s}{a})}\,d\tau \\ &= \frac{1}{a}F(\frac{s}{a}) \end{aligned}$$

II.

$$\frac{1}{s^3+s} = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{B}{s+j} + \frac{C}{s-j}$$

$$A = \left.\frac{1}{s^2+1}\right|_{s=0} = 1$$

$$B = \left.\frac{1}{s(s-j)}\right|_{s=-j} = -\frac{1}{2}$$

$$C = \left.\frac{1}{s(s+j)}\right|_{s=j} = -\frac{1}{2}$$

Therefore,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-\frac{1}{2}}{s+j} + \frac{-\frac{1}{2}}{s-j} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\{\frac{1}{s} - \frac{s}{s^2+1}\} = \boxed{1-\cos(t)}, t \geq 0$$

III.

$$\mathcal{L}^{-1}\{\frac{1}{s^3+9s}\} = \frac{1}{9}\mathcal{L}^{-1}\{\frac{1/3}{(\frac{s}{3})\left((\frac{s}{3})^2+1\right)}\} = \boxed{\frac{1}{9}\left(1-\cos(3t)\right)}, t \geq 0$$

Problem 3 (6 points)

$$\begin{aligned} h(t) &= h_1(t) \star h_2(t) = 3h_2(t) - 3h_2(t-4) \\ &= 3\cos(2\pi t)\left[u\left(t-\frac{1}{4}\right) - u\left(t-\frac{3}{4}\right)\right] - 3\cos\left(2\pi\left(t-4\right)\right)\left[u\left(t-4-\frac{1}{4}\right) - u\left(t-4-\frac{3}{4}\right)\right] \\ &= 3\cos(2\pi t)\left[u\left(t-\frac{1}{4}\right) - u\left(t-\frac{3}{4}\right) - u\left(t-\frac{17}{4}\right) + u\left(t-\frac{19}{4}\right)\right] \end{aligned}$$

For simplicity, let's find $\mathcal{L}\{\cos(2\pi t)u(t-r)\}$ first.

$$\begin{aligned} \mathcal{L}\{\cos(2\pi t)u(t-r)\} &= \mathcal{L}\{\cos\left(2\pi\left(t-r\right)+2\pi r\right)u\left(t-r\right)\} \\ &= e^{-rs}\mathcal{L}\{\cos\left(2\pi t+2\pi r\right)u\left(t\right)\} \\ &= e^{-rs}\mathcal{L}\{\left[\cos\left(2\pi t\right)\cos\left(2\pi r\right) - \sin\left(2\pi t\right)\sin\left(2\pi r\right)\right]u\left(t\right)\} \\ &= e^{-rs}\left[\mathcal{L}\{\cos\left(2\pi t\right)u\left(t\right)\}\cos\left(2\pi r\right) - \mathcal{L}\{\sin\left(2\pi t\right)u\left(t\right)\}\sin\left(2\pi r\right)\right] \\ &= e^{-rs}\frac{s\cos\left(2\pi r\right) - 2\pi\sin\left(2\pi r\right)}{\mathrm{s}^2+4\pi^2} \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\{h(t)\} &= 3\left[e^{-\frac{s}{4}}\frac{-2\pi\left(1\right)}{s^2+4\pi^2} - e^{-\frac{3s}{4}}\frac{-2\pi\left(-1\right)}{s^2+4\pi^2} - e^{-\frac{17s}{4}}\frac{-2\pi\left(1\right)}{s^2+4\pi^2} + e^{-\frac{19s}{4}}\frac{-2\pi\left(-1\right)}{s^2+4\pi^2}\right] \\ &= \boxed{\frac{6\pi}{s^2+4\pi^2}\left(-e^{-\frac{s}{4}} - e^{-\frac{3s}{4}} + e^{-\frac{17s}{4}} + e^{-\frac{19s}{4}}\right)}\end{aligned}$$

Problem 4 (7 points)

(a) Yes.

$$y\left(t\right) = 2x_0\left(t\right) \star h_0\left(t\right) = 2\left[x_0\left(t\right) \star h_0\left(t\right)\right] = \boxed{2y_0\left(t\right)}$$

(b) Yes.

$$\begin{aligned}y\left(t\right) &= \left[x_0\left(t\right) - x_0\left(t-2\right)\right] \star h_0\left(t\right) = x_0\left(t\right) \star h_0\left(t\right) - x_0\left(t-2\right) \star h_0\left(t\right) \\ &= y_0\left(t\right) - \left[x_0\left(t\right) \star h_0\left(t\right)\right] \star \delta\left(t-2\right) = y_0\left(t\right) - y_0\left(t\right) \star \delta\left(t-2\right) \\ &= \boxed{y_0\left(t\right) - y_0\left(t-2\right)}\end{aligned}$$

(c) Yes.

$$\begin{aligned}y\left(t\right) &= x_0\left(t-2\right) \star h_0\left(t+1\right) = \left[x_0\left(t\right) \star \delta\left(t-2\right)\right] \star \left[h_0\left(t\right) \star \delta\left(t+1\right)\right] \\ &= \left[x_0\left(t\right) \star h_0\left(t\right)\right] \star \left[\delta\left(t-2\right) \star \delta\left(t+1\right)\right] = y_0\left(t\right) \star \delta\left(t-1\right) \\ &= \boxed{y_0\left(t-1\right)}\end{aligned}$$

(d) Yes.

$$y\left(t\right) = x_0'\left(t\right) \star h_0\left(t\right) = \int_{-\infty}^{\infty} \frac{d}{dt}x_0\left(t-\tau\right)h_0\left(\tau\right)d\tau = \frac{d}{dt}\int_{-\infty}^{\infty} x_0\left(t-\tau\right)h_0\left(\tau\right)d\tau = \boxed{y_0'\left(t\right)}$$