Total: 25 points

EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, November 15, 2017

NAME: [redacted] **UID:** [redacted]

This exam has 3 problems, for a total of 25 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	10	10
2	5.5	8
3	5	7
Total	20.5	25

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (10 points) The following questions are not related.

1. (3 points) Consider a system where the output $w(t)$ depends on the input $v(t)$ through the equation: $w(t) = \cos(v(t))$. Is this system time invariant? is it linear? is it stable? Explain why or why not.

$v_1(t) \mapsto w_1(t) = \cos(v_1(t))$

$v_2(t) \mapsto w_2(t) = \cos(v_2(t))$

let $v_3(t) = \alpha v_1(t) + \beta v_2(t)$

$v_3(t) \mapsto w_3(t) = \cos(v_3(t))$

$= \cos(\alpha v_1(t) + \beta v_2(t))$

$= \cos(\alpha v_1(t))\cos(\beta v_2(t)) - \sin(\alpha v_1(t))\sin(\beta v_2(t))$

$\neq \alpha\cos(v_1(t)) + \beta\cos(v_2(t))$

$\boxed{\text{it is not linear}}$

3

let $v_4(t) = v(t - t_0)$

$v_4(t) \mapsto w_4(t) = \cos(v_4(t))$

$= \cos(v(t-t_0))$

$= w(t-t_0)$

$\boxed{\text{Therefore, the system is time invariant}}$

Assume $v_5(t) \leq |A| \quad \forall t$

$v_5(t) \mapsto w_5(t) = \cos(v_5(t))$

since $-1 \leq \cos(x) \leq 1, \quad \forall x$

$w_5(t) = \cos(v_5(t)) \leq 1$

$\boxed{\text{then the system is BIBO stable}}$

2. (3 points) Find the time domain representation for a signal with Fourier Transform:

$$X(\omega) = \cos(a\omega)\sin(b\omega)\text{sinc}\left(\frac{\omega}{2\pi}\right).$$

Hint: You can easily find the inverse Fourier Transform of $e^{jt_0\omega}$ sinc $\left(\frac{\omega}{2\pi}\right)$.

$$\cos(aw) = \frac{e^{jaw} + e^{-jaw}}{2}$$

$$\sin(bw) = \frac{e^{jbw} - e^{-jbw}}{2j}$$

$$X(w) = \frac{e^{jaw} + e^{-jaw}}{2} \cdot \frac{e^{jbw} - e^{-jbw}}{2j} \text{sinc}\left(\frac{w}{2\pi}\right)$$

$$= \frac{1}{4j}\text{sinc}\left(\frac{w}{2\pi}\right)\left(e^{jw(a+b)} - e^{jw(a-b)} + e^{jw(b-a)} - e^{-jw(a+b)}\right)$$

$$= \frac{1}{4j}e^{jw(a+b)}\text{sinc}\left(\frac{w}{2\pi}\right) - \frac{1}{4j}e^{jw(a-b)}\text{sinc}\left(\frac{w}{2\pi}\right) + \frac{1}{4j}e^{jw(b-a)}\text{sinc}\left(\frac{w}{2\pi}\right)$$
$$- \frac{1}{4j}e^{-jw(a+b)}\text{sinc}\left(\frac{w}{2\pi}\right)$$

By FT pair

$$x(t) = A\Pi(t/\tau) \leftrightarrow X(w) = A\tau\,\text{sinc}\left(\frac{w\tau}{2\pi}\right)$$

and time shifting property, linearity property

$$x(t-t_0) \leftrightarrow X(w)e^{-jwt_0}.$$

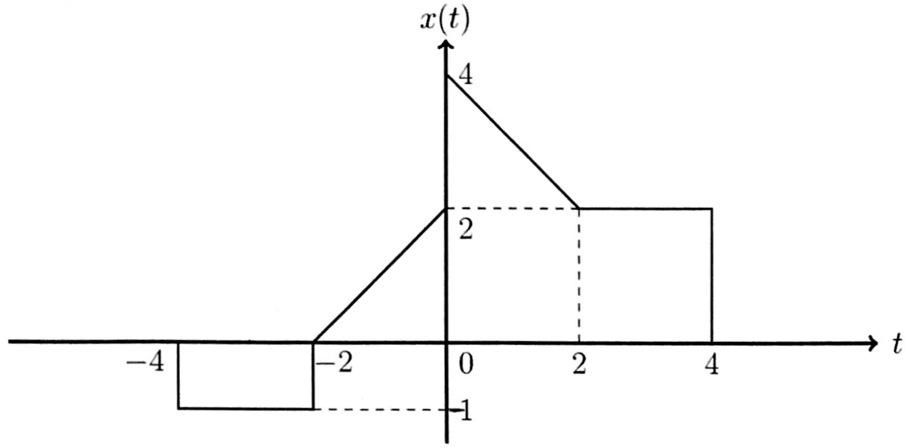
we have

$$x(t) = \frac{1}{4j}\Pi(t+a+b) - \frac{1}{4j}\Pi(t+a-b) + \frac{1}{4j}\Pi(t+b-a) - \frac{1}{4j}\Pi(t-a-b)$$

$$= \frac{1}{4}j\left(-\Pi(t+a+b) + \Pi(t+a-b) - \Pi(t+b-a) + \Pi(t-a-b)\right)$$

3

3. (4 points) Find the Fourier transform of the signal depicted in the following figure.



Hint: You can express $x(t)$ as a linear combination of other signals for which you know the Fourier transform pair.

Assume $a_1 > 0$

$$-\Pi(a_1 t + b_1) = \begin{cases} -1, & |a_1 t + b_1| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} -1, & -\frac{1}{2} \le a_1 t + b_1 \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} -1, & \frac{-\frac{1}{2} - b_1}{a_1} \le t \le \frac{\frac{1}{2} - b_1}{a_1} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} \frac{-\frac{1}{2} - b_1}{a_1} = -4 \\ \frac{\frac{1}{2} - b_1}{a_1} = -2 \end{cases} \rightarrow \begin{cases} -\frac{1}{2} - b_1 = -4a_1 \\ \frac{1}{2} - b_1 = -2a_1 \end{cases} \rightarrow 2a_1 = 1 \rightarrow \begin{matrix} a_1 = \frac{1}{2} \\ b_1 = \frac{3}{2} \end{matrix}$$

$$-\Pi\left(\frac{1}{2}t + \frac{3}{2}\right)$$

Assume $a_2 > 0$

$$2\Pi(a_2 t + b_2) = \begin{cases} 2, & \frac{-\frac{1}{2} - b_2}{a_2} \le t \le \frac{\frac{1}{2} - b_2}{a_2} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} \frac{-\frac{1}{2} - b_2}{a_2} = 0 \\ \frac{\frac{1}{2} - b_2}{a_2} = 4 \end{cases} \rightarrow \begin{cases} b_2 = -\frac{1}{2} \\ \frac{1}{2} - b_2 = 4a_2 \end{cases} \rightarrow \begin{cases} a_2 = \frac{1}{4} \\ b_2 = -\frac{1}{2} \end{cases}$$

$$2\Pi\left(\frac{1}{4}t - \frac{1}{2}\right)$$

$$x(t) = -\Pi\left(\frac{1}{2}t + \frac{3}{2}\right) + 2\Lambda\left(\frac{t}{2}\right) + 2\Pi\left(\frac{1}{4}t - \frac{1}{2}\right)$$

$$= -\Pi\left(\frac{1}{2}(t+3)\right) + 2\Lambda\left(\frac{t}{2}\right) 4 + 2\Pi\left(\frac{1}{4}(t-2)\right)$$

By using linearity property,

$$X(w) = -2\,\text{sinc}\left(\frac{w}{\pi}\right)e^{j3w} + 4\,\text{sinc}^2\left(\frac{w}{\pi}\right) + 8\,\text{sinc}\left(\frac{2w}{\pi}\right)e^{-j2w}$$

4

Problem 2 (8 points) For the following questions, you do not need to do one to proceed with the next - you can use the statements of the previous questions as facts if you need them. **Furthermore, please answer the following questions without using Fourier Series or Fourier transform.**

1. (3 points) Prove the following property of the derivative of convolution, where $\star$ stands for convolution.

$$\frac{d}{dt}\left(f(t) \star g(t)\right) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

Hint: Recall, that the differentiator system, that takes as input a signal and outputs its derivative, is an LTI system. You can use this without proving it.

$$\frac{d}{dt}(f(t) * g(t)) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} f(\tau) \frac{d}{dt}[g(t-\tau)]\, d\tau .$$ ← for the derivative wrt t, τ is constant

$$= f(t) * \left[\frac{d}{dt} g(t)\right]$$ → say this is $r(t)$.

$$r(t-\tau) = \frac{d\, g(t-\tau)}{d(t-\tau)}$$

By the property of convolution we have

$$\frac{d}{dt}(f(t) * g(t)) = \frac{d}{dt}(g(t) * f(t))$$

$$= \frac{d}{dt}\left[\int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} g(\tau) \cdot \frac{d}{dt}[f(t-\tau)]\, d\tau$$

$$= g(t) * \left(\frac{d}{dt}[f(t)]\right)$$

$$= \frac{d}{dt}[f(t)] * g(t)$$

2.5

✓

Therefore

$$\frac{d}{dt}(f(t) * g(t)) = \left(\frac{d}{dt}(f(t))\right) * g(t) = f(t) * \left(\frac{d}{dt} g(t)\right)$$

2. (3 points) For the two signals $x_1(t) = \Pi(\frac{t}{2})$, $x_2(t) = e^{-5|t|}$, find the derivative of the convolution (in the time domain) $z(t) = x_1(t) \star x_2(t)$, that is, find $\frac{d}{dt}z(t)$.

$$\frac{d}{dt} z(t) = \frac{d}{dt}[x_1(t) * x_2(t)]$$

$$= \frac{d}{dt}[\Pi(\tfrac{t}{2}) * e^{-5|t|}]$$

$$= \frac{d}{dt}(\Pi(\tfrac{t}{2})) * e^{-5|t|}$$

$$= \Pi(\tfrac{t}{2}) * \frac{d}{dt} e^{-5|t|}$$

$$= -5(\Pi(\tfrac{t}{2}) * e^{-5|t|})$$

$$= -5\, z(t).$$

fur attempt

!

$$\ln|z(t)| = -5t + c$$

$$z(t) = e^{-5t} k$$

$$\frac{d}{dt}[z(t)] = -5ke^{-5t}.$$

$$\frac{d}{dt} e^{-5|t|}$$

$$= \begin{cases} -5e^{-5t} & t > 0 \\ 5e^{5t} & t < 0 \end{cases}$$

$$t \quad -5e^{-5|t|}$$

3. (2 points) Consider an LTI system, and assume that when the input is $4u(t-1)$ the output is $\cos^2(t)$. Find the impulse response (that is the response to $\delta(t)$) of this system?

Hint: You can use the differentiation property in Question 1.

let $x(t) = 4u(t-1)$

$y(t) = \cos^2(t)$

let the impulse response be $h(t)$,

$y(t) = x(t) * h(t)$

$= \cos^2 t$

then $\frac{d}{dt}[y(t)] = \frac{d}{dt}[x(t) * h(t)]$

$= (\frac{d}{dt} x(t)) * h(t)$

$= \frac{d}{dt}[\cos^2 t]$

$= 2(-\sin t)(\cos t)$

$= -2 \sin t \cos t$

$\frac{d}{dt}(x(t)) = \frac{d}{dt}(4u(t-1)) = 4 \frac{d}{dt}(u(t-1)) = 4\delta(t-1)$

$4\delta(t-1) * h(t) = 4h(t-1) = -2\sin t \cos t$

$h(t-1) = \frac{-\sin t \cos t}{2}$

then $h(t) = \frac{-\sin(t+1)\cos(t+1)}{2}$ ✓

2

5/7

Problem 3 (7 points) Consider a periodic signal $x(t)$, that has the power spectrum depicted on Fig. 1, where C_k is the coefficient of $e^{\frac{j2\pi kt}{T}}$ in the Fourier Series expansion of $x(t)$. Recall that in this plot, because $e^{\frac{j2\pi kt}{T}}$ has the frequency of $\frac{k}{T}$, we associate the magnitude square $|C_k|^2$ with the frequency $\frac{k}{T}$. The following questions are not related to each other.

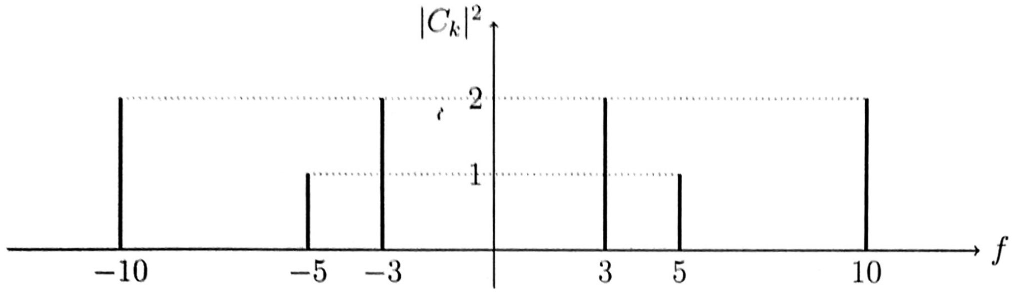


Figure 1: Power Spectrum of $x(t)$.

1. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is real and even. Is there a unique $x(t)$ that has this power spectrum ? Explain why or why not.

Since $x(t)$ is real and even

the C_k is also real and even

$|C_{-3}|^2 = |C_3|^2 = 2 \rightarrow C_{-3} = C_3 = \pm\sqrt{2}$.

$|C_{-5}|^2 = |C_5|^2 = 1 \rightarrow C_{-5} = C_5 = \pm 1$

$|C_{-10}|^2 = |C_{10}|^2 = 2 \rightarrow C_{-10} = C_{10} = \pm\sqrt{2}$.

✓

then, there are more than 1 $x(t)$ that has its power specturm.

$x_1(t) = \sqrt{2}\left(e^{j\frac{6\pi t}{T}} + e^{-j\frac{6\pi t}{T}}\right) + \left(e^{j\frac{10\pi t}{T}} + e^{-j\frac{10\pi t}{T}}\right) + \sqrt{2}\left(e^{j\frac{20\pi t}{T}} + e^{-j\frac{20\pi t}{T}}\right)$

$x_2(t) = -\sqrt{2}\left(e^{j\frac{6\pi t}{T}} + e^{-j\frac{6\pi t}{T}}\right) - \left(e^{j\frac{10\pi t}{T}} + e^{-j\frac{10\pi t}{T}}\right) - \sqrt{2}\left(e^{j\frac{20\pi t}{T}} + e^{-j\frac{20\pi t}{T}}\right)$

2. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is the input to an LTI system. Is it possible that the power spectrum of $y(t)$ (the response to $x(t)$), is the one depicted in Fig. 2? Explain why or why not.

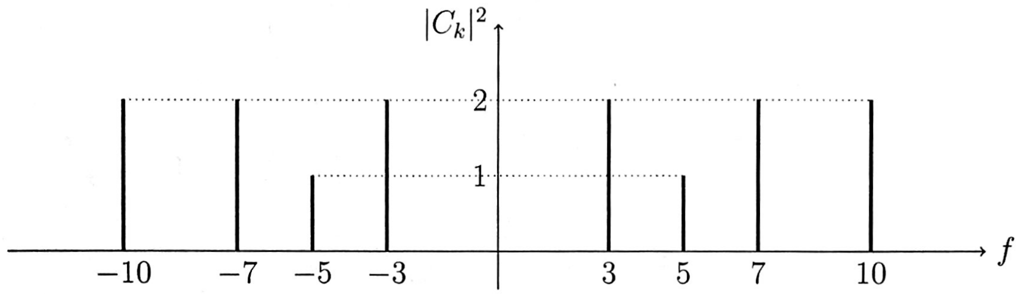


Figure 2: Power Spectrum of $y(t)$.

$Y(t) = x(t) * h(t)$ where $h(t)$ is the impulse response

$$= \sum_{k=-\infty}^{\infty} C_k e^{\frac{j2\pi k t}{T}} * h(t)$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{\frac{j2\pi k\tau}{T}} h(t-\tau)\, d\tau .$$

$$= \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty} e^{\frac{j2\pi k\tau}{T}} h(t-\tau)\, d\tau .$$

we can see that

$C_{-7} = C_7 = 0$ from the previous question.

then in $y(t)$

$C_7 = C_7 = 0$

Therefore

it is not possible to have this power spectrum. ✓

3. (3 points) What is the fundamental period T for the signal $x(t)$ with power spectrum in Fig. 1? Explain your answer.

$$x_1(t) = \sqrt{2}\left(e^{j\frac{6\pi t}{T}} + e^{-j\frac{6\pi t}{T}}\right) + \left(e^{j\frac{10\pi t}{T}} + e^{-j\frac{10\pi t}{T}}\right)$$

$$+ \sqrt{2}\left(e^{j\frac{20\pi t}{T}} + e^{-j\frac{20\pi t}{T}}\right)$$

$$= 2\sqrt{2}\cos\left(\frac{6\pi t}{T}\right) + 2\cos\left(\frac{10\pi t}{T}\right)$$

$$+ 2\sqrt{2}\cos\left(\frac{20\pi t}{T}\right).$$

$\underline{LCM}(6\pi, 10\pi, 20\pi)$

$= 60\pi.$

then $T = 60\pi$ XX