Problem 1 (7 points) The following three questions are not related to each other.

1. (1.5 points) Consider the following signals: $x_1(t) = \text{sinc}(t)$, $x_2(t) = r(t) - 5 + r(-t)$, and $x_3(t) = te^{-3|t|}$. Which of these signals are even? which are odd?

2. (2.5 points) Determine whether the system

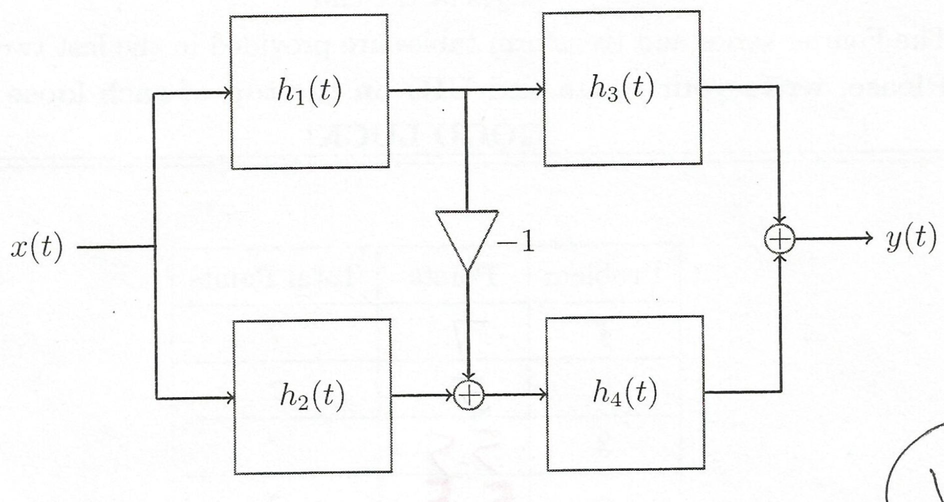
$$y(t) = \begin{cases} x(t-5) & \text{if} \quad |x(t)| \leq B \\ A|x(t)| & \text{otherwise} \end{cases},$$

where $|x(t)|$ is the magnitude of the input $x(t)$, is

   (a) Causal

   (b) Time invariant

3. (3 points) You are are told that the four blocks in the following block diagram represent LTI systems. Determine the expression for the impulse response of the overall system in terms of the impulse responses of the individual systems.



1.5

1. $X_1(-t) = \frac{\sin(-t)}{-t} = \frac{-\sin t}{-t} = \frac{\sin t}{t} = X_1(t)$ $\therefore$ $X_1(t)$ is even

$X_2(-t) = r(-t) - 5 + r(t) = r(t) - 5 + r(-t) = X_2(t)$ $\therefore$ $X_2(t)$ is even

$X_3(-t) = -te^{-3|-t|} = -te^{-3|t|} = -X_3(t)$ $\therefore$ $X_3(t)$ is odd.

2. (a) It is causal since $t > t-5 \ \forall t \in \mathbb{R}$ Not complete

2.5

(b) let $X_1(t) = X(t-t_0)$

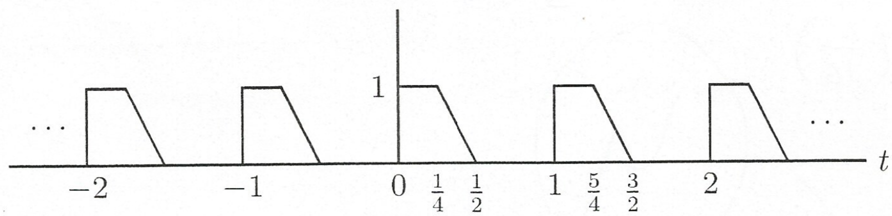
$Y_1(t) = \begin{cases} X_1(t-5) & \text{if } |X(t)| \leq B \\ A|X(t)| & \text{otherwise} \end{cases} = \begin{cases} X(t-t_0-5) & \text{if } |X(t-t_0)| \leq B \\ A|X(t-t_0)| & \text{otherwise} \end{cases} = Y(t-t_0)$

It's time invariant.

Problem 2 (7 points) (a) (4 points) Consider the signal in the following figure, that has period T=1. Calculate the Fourier Series coefficients $\{c_k\}$.

$m = \frac{1-0}{\frac{1}{4}-\frac{1}{2}} = \frac{1}{-\frac{1}{4}} = -4$

$y = -4(t-\frac{1}{2}) \Rightarrow y = -4t+2$



(b) (3 points) As we have discussed, the signal in the previous question also has as period all integer multiples of T, for instance, T=10 is also a period. What will happen if you calculate the Fourier Series coefficients, for the signal in part (a), assuming that T=10? Could you directly tell what these coefficients would be from the coefficients $\{c_k\}$ you calculated in part (a)?

a) $C_k = \frac{1}{T}\int_0^T x(t)e^{-i\frac{2\pi k}{T}t}dt = \int_0^{\frac14} e^{-i2\pi kt}dt + \int_{\frac14}^{\frac12}(-4t+2)e^{-i2\pi kt}dt$

$k \neq 0$

$= \frac{-1}{i2\pi k}\left[e^{-i2\pi kt}\right]_0^{\frac14} - 4\int_{\frac14}^{\frac12} te^{-i2\pi kt}dt + 2\int_{\frac14}^{\frac12} e^{-i2\pi kt}dt$

$= \frac{-1}{i2\pi k}\left[e^{-\frac{i2\pi k}{4}} - 1\right] - 4\left[\frac{t}{-i2\pi k}e^{-i2\pi kt}\Big|_{\frac14}^{\frac12} + \int_{\frac14}^{\frac12}\frac{1}{i2\pi k}e^{-i2\pi kt}dt\right] + \frac{-2}{i2\pi k}e^{-i2\pi kt}\Big|_{\frac14}^{\frac12}$

$= \frac{-1}{i2\pi k}\left[e^{-i\frac{\pi}{2}k} - 1\right] - 4\left[\frac{t}{-i2\pi k}e^{-i2\pi kt} - \frac{1}{(i2\pi k)^2}e^{-i2\pi kt}\right]_{\frac14}^{\frac12} - \frac{2}{i2\pi k}\left[e^{-i\pi k} - e^{-i\frac{\pi}{2}k}\right.$

$= \frac{-1}{i2\pi k}\left[e^{-i\frac{\pi}{2}k} - 1 + 2e^{-i\pi k} - 2e^{-i\frac{\pi}{2}k}\right] + 4\left[\frac{\frac12}{i2\pi k}e^{-i\pi k} + \frac{1}{(i2\pi k)^2}e^{-i\pi k} - \left(\frac{\frac14}{i2\pi k}e^{-i\frac{\pi}{2}k} - \frac{e^{-i\frac{\pi}{2}k}}{(i2\pi k)^2}\right.\right.$

$= \frac{-1}{i2\pi k}\left[2e^{-i\pi k} - e^{-i\frac{\pi}{2}k} - 1\right] + \frac{2}{i2\pi k}e^{-i\pi k} + \frac{4e^{-i\pi k}}{(i2\pi k)^2} - \frac{e^{-i\frac{\pi}{2}k}}{i2\pi k} + \frac{4e^{-i\frac{\pi}{2}k}}{(i2\pi k)^2}$

$= \frac{1}{i2\pi k} + \frac{4(-1)^k}{(i2\pi k)^2} + \frac{4(-i)^k}{(i2\pi k)^2}$

$= \frac{-4(-1)^k}{(2\pi k)^2} - \frac{4(-i)^k}{(2\pi k)^2} - \frac{i}{2\pi k}$

$$C_0 = \int_0^{\frac{1}{4}} dt + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4t+2)\, dt$$

$$= \frac{1}{4} + 2\left(\frac{1}{2}-\frac{1}{4}\right) - 2t^2\Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{1}{4} + 2\left(\frac{1}{4}\right) - 2\left[\frac{1}{4}-\frac{1}{16}\right]$$

$$= \frac{3}{4} - 2\left(\frac{3}{16}\right)$$

$$= \frac{6}{8} - \frac{3}{8}$$

$$= \frac{3}{8}$$

✓ 4

$$C_k = \begin{cases} \frac{3}{8} & , k=0 \\ \frac{4(-1)^{k+1}}{(2\pi k)^2} - \frac{4(-i)^k}{(2\pi k)^2} - \frac{i}{2\pi k} & \end{cases}$$

b) $\{C_k\}$ will not change even if T changes from 1 to 10.

$$C_k = \frac{1}{T}\int_0^T x(t) e^{-\frac{i2\pi k t}{T}} dt$$

1

$$= \frac{-\cancel{T}}{\cancel{T}\, i2\pi k} x(t)\left[e^{-\frac{i2\pi k \cancel{T}}{\cancel{T}}} - 1\right] - \int_0^T \frac{-1}{i2\pi k} x'(t) e^{-\frac{i2\pi k t}{T}} dt$$

The Ts cancel out.

What about this?

You need to put limits of t into x(t) as well.

X

look at solutions.

$$x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0' t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\frac{\omega_0}{10} t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad d_k = \begin{cases} C_{\frac{k}{10}} & k = 10n \\ 0 & \text{otherwise} \end{cases}$$

Problem 3 (7 points)

1. (2 points) Calculate the Fourier transform of the signal $\Pi(at-b)$, where $\Pi(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & \text{else} \end{cases}$ as a function of the parameters a and b.

$a=\frac{1}{2}$ $\frac{5}{2}=b$ $5=\frac{b}{a}$

2. (2 points) Let $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ be the derivative of the signal x(t), where $\Pi(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & \text{else} \end{cases}$. Calculate the Fourier transform of x(t).

3. (3 points) Calculate the following integral

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau$$

1. $\Pi(t) \longleftrightarrow \frac{\sin(\frac{w}{2})}{\frac{w}{2}}$ $\quad \Pi(t-\frac{b}{a}) \longleftrightarrow e^{-i\frac{b}{a}w} \operatorname{sinc}(\frac{w}{2})$

1

$$\Pi(at-b) \longleftrightarrow \boxed{\frac{1}{|a|} e^{-i\frac{b}{a^2}w} \operatorname{sinc}(\frac{w}{2a})} \times$$

If $y(t)=\Pi(t-\frac{b}{a}) \to$ ~~that~~ $y(at)=\Pi(at-\frac{b}{a})$. time scaling is only for the t term.

2. $\Pi(t-2) - \frac{1}{2}\Pi(\frac{t}{2}-\frac{5}{2}) \longleftrightarrow e^{-i2w} \operatorname{sinc}(\frac{w}{2}) - \cancel{\frac{1}{2}}\cancel{2} e^{-i\frac{5}{\cancel{2}}\cancel{2}w} \operatorname{sinc}(w)$

2

$$= e^{-i2w} \operatorname{sinc}(\frac{w}{2}) - e^{-i5w} \operatorname{sinc}(w) = Y(w)$$

$$y(t) = \frac{dx}{dt} \longleftrightarrow Y(w) = iwX(w)$$

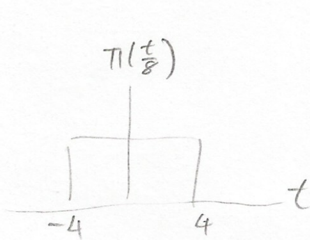
$$\Rightarrow X(w) = \frac{1}{iw} Y(w) = \boxed{\frac{1}{iw}\left[e^{-i2w} \operatorname{sinc}(\frac{w}{2}) - e^{-i5w} \operatorname{sinc}(w)\right]}$$

3. $\int_{-\infty}^{\infty} \frac{248}{4} \frac{\sin(18\tau)\sin(4\tau)}{18\tau \cdot 4\tau} e^{i11\tau} d\tau = 2\pi \cdot \frac{1}{4} z(11)$ +1 -0.5

2.5

let $X_1(w) = \cancel{36}^{18} \operatorname{sinc}(18w)$ & $X_2(w) = \cancel{8}^{4} \operatorname{sinc}(4w)$ 36 π t

$z(t) = X_1(t) * X_2(t) = \Pi(\frac{t}{36}) * \Pi(\frac{t}{8})$ +1



$$2\pi \cdot \frac{1}{4} z(11) = \frac{\pi}{2}\left[\Pi(\frac{11}{36}) * \Pi(\frac{11}{8})\right] = \frac{\pi}{2}\left[\Pi(\frac{11}{36}) * 0\right] = 0$$

+1

5.~~5~~/7

Problem 4 (7 points) Sometimes we work with systems that take as input two signals, say $f(t)$ and $g(t)$ and produce at their output one signal, say $y(t)$. One way of analyzing such systems is by assuming they take as input a 2×1 vector $x(t)$ that has as elements the signals $f(t)$ and $g(t)$; we then apply the definitions for linearity and time invariance on the vector input $x(t)$.

Consider a system that takes as input two real signals $f(t)$ and $g(t)$ and calculates as output their inner product $y(t)$ defined as

$$y(t) = (f, g) = \int_{-\infty}^{\infty} f(t)g(t)\,dt$$

Recall that the time reverse of a signal $x(t)$ is the signal $x(-t)$, and the time shifted version of $x(t)$ by some constant t_0 is the signal $x(t - t_0)$.

(a) If both $f(t)$ and $g(t)$ are time reversed, what happens to their inner product?

(b) Assume that only one of $f(t)$ and $g(t)$ is time reversed, does the outcome depend on which one was reversed or no?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) Assume that you can use as blocks the following systems: a block that takes as input a signal and time reverses it, a block that takes as input 2 signals $x_1(t)$ and $x_2(t)$ and outputs the signal $y(t) = x_1(t) * x_2(t)$ that is their convolution, a block that takes as input a signal and delays it by a fixed amount t_0 we can select, and a block that takes as input a signal $x(t)$ and outputs the constant value $\int_{-\infty}^{\infty} x(t)\delta(t - t_1)dt$ for a constant t_1 we can select. Can you connect (some of) these blocks to create a system that takes as input two signals and outputs their inner product value?

(e) Is the system that implements the inner product time invariant? Is it linear?

a) $\int_{-\infty}^{\infty} f(-t)g(-t)\,dt$ let $\lambda = -t$ $\Rightarrow \int_{\infty}^{-\infty} f(\lambda)g(\lambda)(-d\lambda) = \int_{-\infty}^{\infty} f(\lambda)g(\lambda)d\lambda$

$= y(\lambda) = y(t)$ $\boxed{\text{No change}}$ ✓

b) Yes, if the time reversed function is even,

$\int_{-\infty}^{\infty} f(-t)g(t)\,dt = \int_{-\infty}^{\infty} f(t)g(t)dt = y(t)$

if the time reversed function is odd,

$\int_{-\infty}^{\infty} f(-t)g(t)\,dt = -\int_{-\infty}^{\infty} f(t)g(t)dt = -y(t)$

if neither, $\int_{-\infty}^{\infty} f(-t)g(t)\,dt \neq y(t) \neq -y(t)$

c) $\int_{-\infty}^{\infty} f(t-t_0)g(t-t_0)\,dt$ let $\lambda = t-t_0$

$\lambda + t_0 = t$

$t \to \infty \quad \lambda + t_0 \to \infty$

$t \to -\infty \quad \lambda + t_0 \to -\infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(\lambda)g(\lambda)\,d\lambda = y(\lambda)$$

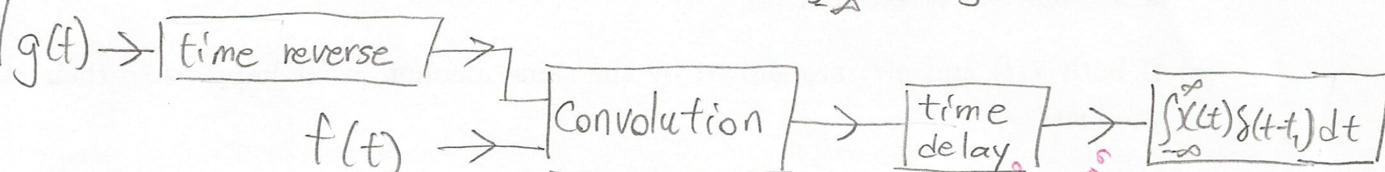
$$= y(t) \quad \boxed{\text{No change}}$$

d) $g(t) * f(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$ $\qquad \int_{-\infty}^{\infty} f(\tau)g(-(t-t_0)+\tau)\,d\tau$

$g(-t) * f(t) = \int_{-\infty}^{\infty} f(\tau)g(-t+\tau)\,d\tau$ $\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(-t+t_0+\tau)\,d\tau\,\delta(\lambda-t_0)\,d$

$$= \int_{-\infty}^{\infty} f(\tau)g(\tau)\,d\tau$$

$\boxed{\text{Yes}}$



R, why? you can use $t_1 = 0$

e) It is not linear. We have two input function

$$\int_{-\infty}^{\infty} [\alpha f_1(t) + \beta f_2(t)][\alpha g_1(t) + \beta g_2(t)]\,dt \neq \alpha\int_{-\infty}^{\infty} f_1(t)g_1(t)\,dt + \beta\int_{-\infty}^{\infty} f_2(t)g_2(t)\,d$$

It is not time invariant.

from (c), we know if $f(t)$ and $g(t)$ is shifted by the same amount, the output doesn't change. $y(t) \neq y(t-t_0)$.

?

$y(t)$ is constant ! !