13

$1-4\left(\frac{2\sqrt{10}j}{7\sqrt{10}j-}\right)$

$\frac{12s}{s^2+10}$

$\frac{-10-5j\sqrt{10}+6}{-10+7j\sqrt{10}+10}$

$\frac{-4-5j\sqrt{10}}{7j\sqrt{10}}$

$-12s-4$

$\frac{4j}{7\sqrt{10}}-\frac{5}{7}$

Problem 1 (15 pts)
Given the following transfer function of an LTI system

$$H(s) = \frac{s^2 - 5s + 6}{s^2 + 7s + 10}$$

$-12s = 7s$ $+4+10$

(a) (5 pts) Sketch pole-zero plot of $H(s)$. State whether the system is BIBO stable or not. Explain.

(b) (5 pts) Compute the inverse Laplace transform to find a causal impulse response function $h(t)$.

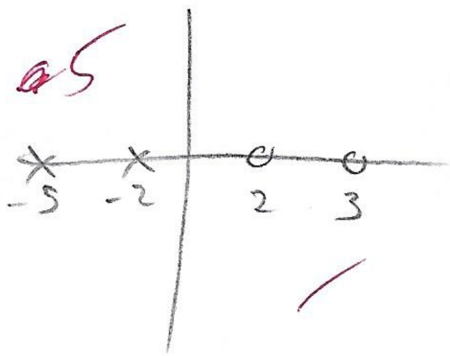
(c) (5 pts) Find the steady state output $y(t)$ when the input is $x(t) = 12\cos(\sqrt{10}\,t)$.

a) $(s+5)(s+2)=0$ $\quad s=-5,-2$

$s^2+7s+10$ $\quad (s-2)(s-3)=0$ $\quad s=2,3$

s^2-5s+6

5



BIBO stable since all poles are on LHS, ROC contains Im axis

b) $1 + \frac{-12s-4}{(s+5)(s+2)}$ $\quad -4(3s+1)$

$As+Bs=3s$

$2A+5B=1$

$3B=-5 \quad B=-\frac{5}{3}$

$A=\frac{14}{3}$

$\frac{A}{s+5}+\frac{B}{s+2}=\frac{-12s-4}{(s+5)(s+2)}$

2

$As+2A+Bs+5B=-12s-4$

$A+B=-12$ $\quad B=\frac{20}{3}$ $\quad A=-\frac{36}{3}-\frac{20}{3}=-\frac{56}{3}$

$2A+5B=-4$

$3B=20$

$h(t)=\delta(t)-\frac{56}{3}e^{-5t}u(t)-\frac{20}{3}e^{-2t}u(t)$

$+\frac{20}{3}e^{-2t}$

15

Problem 2 (15 pts)
The input-output relationship for an LTI system is given by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 5x(t), t \geq 0$$

with initial conditions $y'(0) = y(0) = 0, x(0) = 0$.

(a) (5 pts) Find the transfer function $H(s)$.

(b) (5 pts) Find the impulse response function $h(t)$.

(c) (5 pts) Find the output $y(t)$ when the input is $x(t) = e^{-5(t-4)}u(t-4)$.

a) $s^2 Y(s) + 2sY(s) + 5Y(s) = sX(s) + 5X(s)$

$(s^2+2s+5)Y(s) = (s+5)X(s)$

5 $H(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{s^2+2s+5}$ $\quad s^2+2s+1+4$

b) $s_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{s+5}{(s+1)^2+(2)^2}$

5

$\frac{s+1}{(s+1)^2+(2)^2} + \frac{4}{(s+1)^2+(2)^2}$

$h(t) = e^{-t}\cos(2t)u(t) + 2e^{-t}\sin(2t)u(t)$

c) $Y(s) = X(s)H(s) = \frac{e^{-4s}}{s+5} \cdot \frac{s+5}{s^2+2s+5} = \frac{e^{-4s}}{s^2+2s+5} \rightarrow$

$\cos(2\pi) + j\sin(2\pi)$

$1 \quad + 0$

1

18

Problem 3 (18 pts)
Consider an LTI system with impulse response function

$$h(t) = e^{-\pi(t-2)}u(t-2).$$

When a periodic signal $x(t)$ with period 2 is applied to this system, we get the following periodic output

$$y(t) = 1 + \cos(\pi t) + \sin(3\pi t),$$

(a) (10 pts) Find the exponential Fourier series coefficients of the input $x(t)$.

(b) (8 pts) Sketch magnitude and phase spectra of X_k.
Hint: Use the following while plotting phase spectrum: $tan^{-1}(1) = \frac{\pi}{4}$, $tan^{-1}(-1) = -\frac{\pi}{4}$, $tan^{-1}(\frac{1}{3}) \approx \frac{\pi}{10}$, *and* $tan^{-1}(-\frac{1}{3}) \approx -\frac{\pi}{10}$.

a) $\omega = \pi$ $\quad H(s) = \frac{e^{-2s}}{s+\pi}$ ✓

$Y_K = 1e^{j0} + \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} + \frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-j3\pi t}$

$Y_K = X_K H(jk\omega)$

$X_K = \frac{Y_K}{H(jk\omega)}$ $\quad Y_K = \begin{cases} 1 & k=0 \\ \frac{1}{2} & k=\pm 1 \\ \frac{1}{2j} & k=3 \\ -\frac{1}{2j} & k=-3 \end{cases}$

$H(0) = \frac{1}{\pi}$ ✓ $\quad X_0 = 1 \cdot \pi$

$H(j\pi) = \frac{e^{-2j\pi}}{j\pi+\pi} = \frac{1}{j\pi+\pi}$ $\quad X_1 = \frac{1}{2}(j\pi+\pi)$ →

$-\pi \sin(\frac{\pi}{2}t)$ $\frac{-\pi}{8}\int_0^8 \sin(\frac{\pi}{2}t)e^{-jk\frac{\pi}{4}t}dt$

$\frac{j\pi}{16}\int e^{-j\frac{\pi}{4}(k-2)} - e^{-j\frac{\pi}{4}(k+2)} dt$ 12 $\frac{\pi}{8}\int \frac{1}{2j}(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t})e^{-jk\frac{\pi}{4}t}dt)$

Problem 4 (12 pts) Consider the following periodic signal $x(t)$

$$x(t) = 2\cos(\pi t/2) - \sin(\pi t/4)$$

$T_1 = \frac{2\pi}{\frac{\pi}{2}} = 4$

(a) (6 pts) Find the exponential Fourier series coefficients of $x(t)$.

$T_2 = \frac{2\pi}{\frac{\pi}{4}} = 8$

(b) (6 pts) Find the exponential Fourier series coefficients of

$$y(t) = \frac{dx(t)}{dt}.$$

$T = 8$

$\omega_0 = \frac{\pi}{4}$

a) $x(t) = \underbrace{e^{j\frac{\pi}{2}t}}_{k=2} + \underbrace{e^{-j\frac{\pi}{2}t}}_{k=-2} - \frac{1}{2j}(\underbrace{e^{j\frac{\pi}{4}t}}_{k=1} - \underbrace{e^{-j\frac{\pi}{4}t}}_{k=-1})$

$\frac{\pi}{2} = 2\omega_0$

$$\boxed{X_k = \begin{cases} 1 & k = \pm 2 \\ -\frac{1}{2j} & k = 1 \\ \frac{1}{2j} & k = -1 \end{cases}}$$

b) $Y_k = j\omega_0 k X_k =$

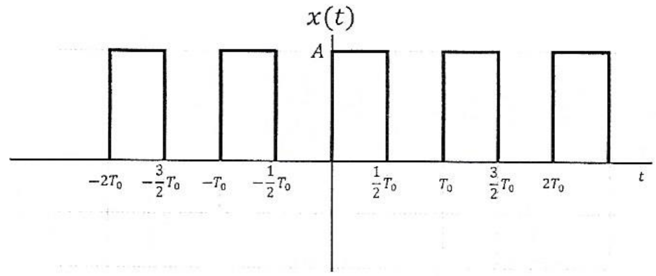
$j\frac{\pi}{4} \cdot -2 \cdot 1 = -j\frac{\pi}{2}$

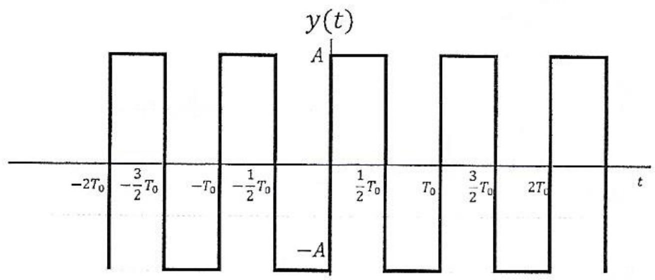
$j\frac{\pi}{4} \cdot -1 \cdot \frac{1}{2j}$

$$\begin{cases} j\frac{\pi}{4}\cdot 2\cdot 1 = j\frac{\pi}{2} & k=2 \\ -j\frac{\pi}{2} & k=-2 \\ j\frac{\pi}{4}\cdot 1\cdot \frac{-1}{2j} = -\frac{\pi}{8} & k=1 \\ -\frac{\pi}{8} & k=-1 \end{cases}$$

15

Problem 5 (15 pts) Consider the following periodic signals $x(t)$ and $y(t)$





(a) (5 pts) Determine the complex exponential Fourier series of $x(t)$, i.e., find the coefficients X_k that satisfy

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 kt}$$

(b) (5 pts) Determine the trigonometric Fourier series of $x(t)$, i.e., find the coefficients a_k and b_k that satisfy

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) - b_k \sin(k\omega_0 t)$$

(c) (5 pts) Using part (a), determine the complex exponential Fourier series of $y(t)$, i.e., find the coefficients Y_k that satisfy

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j\omega_0 kt}$$

$e^{-2t} = e^{-2t}\cdot e^{6}\cdot e^{-6} = e^{-2(t-3)}e^{-6}$

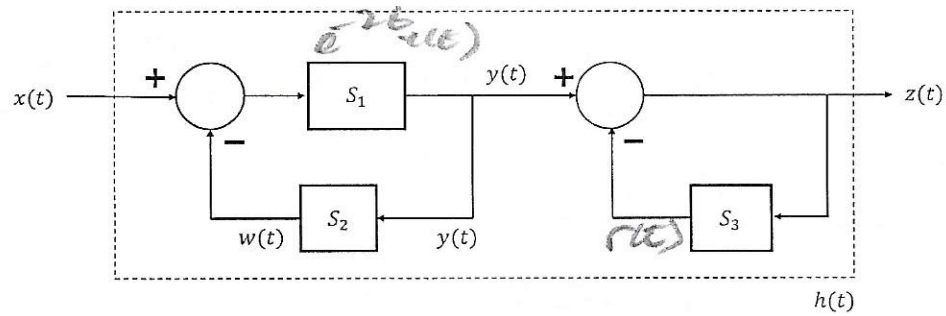
15

Excellent

Problem 6 (15 pts) Consider the following LTI system with input $x(t)$, output $z(t)$, and impulse response function (IRF) $h(t)$. Systems S_1, S_2 and S_3 are LTI systems as well with the following characteristics

1. IRF of S_1 is $h_1(t) = e^{-2t}u(t)$.
2. Input-output relation for S_2 is $w(t) = \frac{d}{dt}y(t)$
3. IRF of S_3 is $h_3(t) = e^{-t}u(t)$.

(Assume all the signals at $t = 0$ are zero, i.e., all initial conditions are zero).



$y(t) - r(t) = z(t)$

$r(t) = z(t)$

(a) (8 pts) Compute the IRF of entire system $h(t)$ and its Laplace transform $H(s)$.

(b) (7 pts) Find the output $z(t)$ if the following input is applied to the system:

$$x(t) = e^{-2t}\cos(8t - 24)u(t - 3) \quad (1)$$

a) $y(t) = (x(t) - w(t)) * S_1(t)$ ✓

$w(t) = y'(t)$ ✓

$y(t) = (x(t) - y'(t)) * S_1(t)$

$z(t) = y(t) - z(t) * S_3(t)$

$Y(s) = (X(s) - sY(s))\frac{1}{s+2}$

$Y(s) + \frac{s}{...}Y(s)$ ...