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UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm Examination II
Date: March 6, 2018
Duration: 1 hr. 50 min.

INSTRUCTIONS:

- The exam has 6 problems and 15 pages.
- The exam is closed-book.
- Two cheat sheets of A4 size are allowed.
- Calculator is NOT allowed.
- Put your discussion session in the top-right corner ↗↗

Your name: [redacted]

Student ID: [redacted]

Table 1: Score Table

Problem	a	b	c	d	Score	
1	5	5	5		15	9
2	5	5	5		15	12
3	10	8			18	16
4	6	6			12	~~10~~ 12
5	5	5	5		15	10
6	8	7			15	8
Total					90	67

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Problem 1 (15 pts)
Given the following transfer function of an LTI system

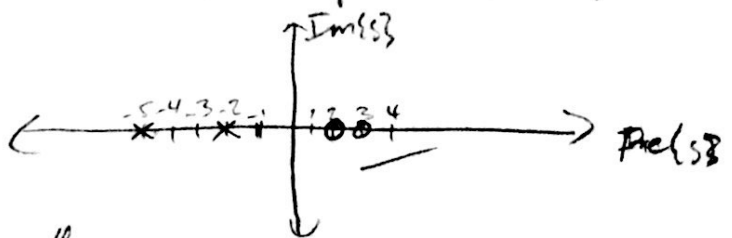
$$H(s) = \frac{s^2 - 5s + 6}{s^2 + 7s + 10}$$

(a) (5 pts) Sketch pole-zero plot of $H(s)$. State whether the system is BIBO stable or not. Explain.

(b) (5 pts) Compute the inverse Laplace transform to find a causal impulse response function $h(t)$.

(c) (5 pts) Find the steady state output $y(t)$ when the input is $x(t) = 12\cos(\sqrt{10}\, t)$.

a) zeros : $s = 2, 3$ poles: $s = -2, -5$



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yes, the system is BIBO stable b/c the $j\Omega$ axis is included in the ROC ($Re\{s\} > -2$)

b) $H(s) = \frac{s^2+7s+10}{s^2+7s+10} + \frac{-7s-10-5s+6}{s^2+7s+10} = 1 + \frac{-12s-4}{s^2+7s+10} = 1 - 4\left(\frac{3s-1}{s^2+7s+10}\right)$

$\frac{3s-1}{s^2+7s+10} = \frac{A}{s+5} + \frac{B}{s+2}$

$A(s+2) + B(s+5) = 3s-1$
$(A+B)s = 3s \Rightarrow A+B = 3$
$2A + 5B = -1$
$-2A - 2B = -6$
$3B = -7$
$B = -\frac{7}{3}$ $A = \frac{16}{3}$

4 so $H(s) = 1 - 4\left(\frac{\frac{16}{3}}{s+5} + \frac{-\frac{7}{3}}{s+2}\right)$

$\boxed{h(t) = \delta(t) - \frac{64}{3} e^{-5t} u(t) + \frac{28}{3} e^{-2t} u(t)}$

c) $X(s) = 12\left(\frac{s}{s^2+10}\right)$ ✗

$Y(s) = X(s)H(s) = 12\left(\frac{s}{s^2+10}\right)\left(1 - \frac{\frac{64}{3}}{s+5} + \frac{\frac{28}{3}}{s+2}\right)$

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Problem 2 (15 pts)
The input-output relationship for an LTI system is given by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 5x(t), t \geq 0$$

with initial conditions $y'(0) = y(0) = 0, x(0) = 0$.

(a) (5 pts) Find the transfer function $H(s)$.

(b) (5 pts) Find the impulse response function $h(t)$.

(c) (5 pts) Find the output $y(t)$ when the input is $x(t) = e^{-5(t-4)}u(t-4)$.

a) $s^2Y(s) - \cancel{sy(0^-)} - \cancel{y'(0^-)} + 2(sY(s) - \cancel{y(0)}) + 5Y(s) = sX(s) - \cancel{x(0)} + 5X(s)$

$(s^2+2s+5)Y(s) = (s+5)X(s)$

5

$H(s) = \frac{Y(s)}{X(s)} = \frac{s+5}{s^2+2s+5}$ ✓

b) $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$H(s) = \frac{s+5}{s^2+2s+5} = \frac{A}{s+1+2j} + \frac{B}{s+1-2j}$ $\quad s = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2j$

$h(t) = Ae^{-(1+2j)t}u(t) + Be^{-(1-2j)t}u(t)$

$A(s+1-2j) + B(s+1+2j) = s+5$

$(A+B)s = s \Rightarrow A+B = 1$

$B = 1-A$

$A(1-2j) + B(1+2j) = 5$

3

$A(1-2j) + (1-A)(1+2j) = 5$

$A - 2jA + 1 + 2j - A - 2jA = 5$

$1 + 2j - 4jA = 5$

$2j - 4jA = 4$

$A = \frac{4-2j}{-4j} = \frac{-2+j}{2j} = \frac{2j+1}{2}$

$B = A^* = \frac{-2j+1}{2}$

×

$h(t) = (j+\frac{1}{2})e^{-(1+2j)t}u(t) + (-j+\frac{1}{2})e^{-(1-2j)t}u(t)$

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i) $Y(s) = X(s)H(s)$ $\qquad e^{-5t}u(t)$

$X(s) = e^{-4s}\dfrac{1}{s+5}$

$Y(s) = e^{-4s}\left(\dfrac{1}{\cancel{s+5}}\right)\left(\dfrac{\cancel{s+5}}{s^2+2s+5}\right) = e^{-4s}\left(\dfrac{1}{(s^2+2s+1)+4}\right)$

$Y(s) = \frac{1}{2}e^{-4s}\left(\dfrac{2}{(s+1)^2+2^2}\right)$ $\qquad \dfrac{2}{(s+1)^2+2^2} \xrightarrow{\mathcal{L}^{-1}} e^{-t}\sin(2t)$

$\boxed{y(t) = \frac{1}{2}e^{-(t-4)}\sin(2(t-4))}\,u(t-4).$

5 d

$$e^{-\pi\lambda}u(\lambda) \xrightarrow{L} \frac{1}{s+\pi}$$

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Problem 3 (18 pts)
Consider an LTI system with impulse response function

$$h(t) = e^{-\pi(t-2)}u(t-2).$$

When a periodic signal $x(t)$ with period 2 is applied to this system, we get the following periodic output

$$y(t) = 1 + \cos(\pi t) + \sin(3\pi t).$$

(a) (10 pts) Find the exponential Fourier series coefficients of the input $x(t)$.

(b) (8 pts) Sketch magnitude and phase spectra of X_k.
Hint: Use the following while plotting phase spectrum: $tan^{-1}(1) = \frac{\pi}{4}$, $tan^{-1}(-1) = -\frac{\pi}{4}$, $tan^{-1}(\frac{1}{3}) \approx \frac{\pi}{10}$, *and* $tan^{-1}(-\frac{1}{3}) \approx -\frac{\pi}{10}$.

$\Omega_0 = \pi$

$$y(t) = 1 + \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) + \frac{1}{2j}\left(e^{j3\pi t} - e^{-j3\pi t}\right)$$

$$y(t) = 1 + \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} - \frac{j}{2}e^{j3\pi t} + \frac{j}{2}e^{-j3\pi t}$$

$Y_0 = 1 \quad Y_1 = Y_{-1} = \frac{1}{2} \quad Y_3 = -\frac{j}{2} \quad Y_{-3} = \frac{j}{2}$

$$H(s) = e^{-2s}\left(\frac{1}{s+\pi}\right)$$

$$H(jk\Omega_0) = e^{-2jk\pi}\left(\frac{1}{jk\pi + \pi}\right) = \frac{1}{\pi(1+jk)}$$ ← never 0, so if $X_k \neq 0$, $Y_k \neq 0$

$Y_k = H(jk\Omega_0)X_k$

$Y_1 = H(j\pi)X_1 = \frac{1}{\pi(1+j)}X_1 = \frac{1}{2}$

$Y_{-1} = \frac{1}{\pi(1-j)}X_{-1} = \frac{1}{2}$

$Y_0 = \frac{1}{\pi}X_0 = 1$

$Y_3 = \frac{1}{\pi(1+3j)}X_3 = \frac{-j}{2}$

$Y_{-3} = \frac{1}{\pi(1-3j)}X_3 = \frac{j}{2}$

$X_1 = \frac{\pi(1+j)}{2}$

$X_{-1} = \frac{\pi(1-j)}{2}$

$X_0 = \pi$

$X_3 = \frac{-j\pi(1+3j)}{2}$

$X_3 = \frac{j\pi(1-3j)}{2}$

$X_k = 0$ o/w

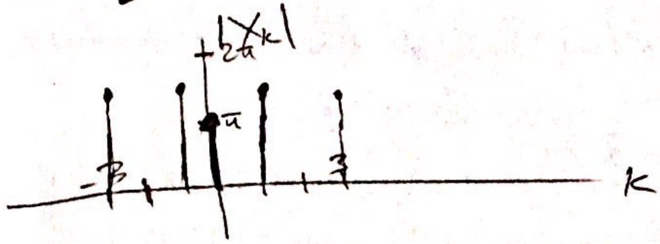
b) $|X_1| = \frac{\pi}{2}\sqrt{1^2+1^2} = \pi\frac{\sqrt{2}}{2}$

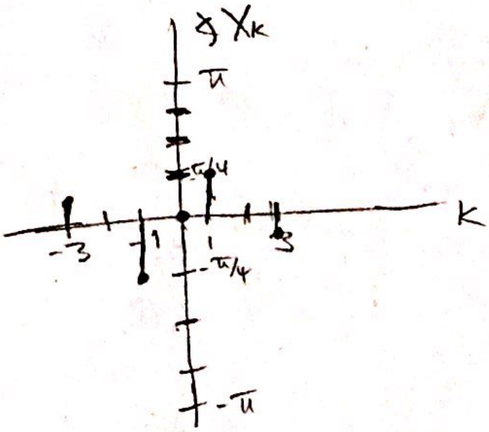
$|X_{-1}| = \pi\frac{\sqrt{2}}{2}$

$|X_3| = \left|\frac{-j\pi+3}{2}\right| = \frac{1}{2}\sqrt{\pi^2+3^2}$ -2

$|X_{-3}| = \frac{1}{2}\sqrt{\pi^2+3^2}$

$\frac{\sqrt{18}}{2}$ $\frac{3\sqrt{2}}{2}$





$\measuredangle X_k = \tan^{-1}\left(\frac{Im\{X_k\}}{Re\{X_k\}}\right)$

$\measuredangle X_0 = \tan^{-1}(0) = 0$

$\measuredangle X_1 = \tan^{-1}(1) = \frac{\pi}{4}$

$\measuredangle X_{-1} = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\measuredangle X_3 = \tan^{-1}\left(-\frac{1}{3}\right) \approx -\frac{\pi}{10}$

$\measuredangle X_{-3} = \tan^{-1}\left(\frac{1}{3}\right) \approx \frac{\pi}{10}$

12 **Problem 4** (12 pts) Consider the following periodic signal $x(t)$

$$x(t) = 2\cos(\pi t/2) - \sin(\pi t/4)$$

$T_1 = 4$ $\quad T_2 = 8$ $\quad T_0 = 8$ $\quad \Omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{4}$

(a) (6 pts) Find the exponential Fourier series coefficients of $x(t)$.

(b) (6 pts) Find the exponential Fourier series coefficients of

$$y(t) = \frac{dx(t)}{dt}.$$

a) $x(t) = e^{j\pi t/2} + e^{-j\pi t/2} - \frac{1}{2j}\left(e^{j\pi t/4} - e^{-j\pi t/4}\right)$

$x(t) = e^{j\pi t/2} + e^{-j\pi t/2} + \frac{j}{2} e^{j\pi t/4} - \frac{j}{2} e^{-j\pi t/4}$

$X_1 = \frac{j}{2}$ $\quad X_{-1} = -\frac{j}{2}$ $\quad X_2 = 1$ $\quad X_{-2} = 1$

$X_k = 0$ o/w

b) $Y_k = jk\Omega_0 X_k$

$y(t) = \frac{d}{dt}\left(\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}\right)$

$y(t) = \sum_{k=-\infty}^{\infty} X_k \frac{d}{dt}\left(e^{jk\omega_0 t}\right)$

$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{jk\omega_0 X_k}_{Y_k} e^{jk\omega_0 t}$

So $Y_1 = \frac{j\pi}{4} \cdot \frac{j}{2} = -\frac{\pi}{8}$

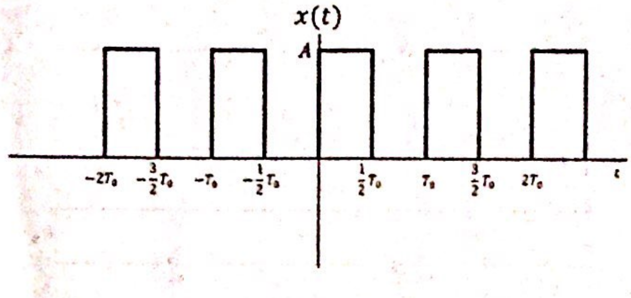
$Y_{-1} = \frac{-j\pi}{4} \cdot \frac{-j}{2} = -\frac{\pi}{8}$

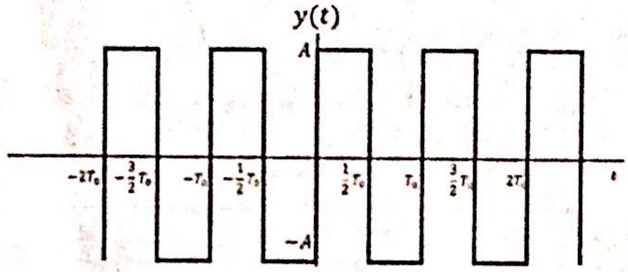
$Y_2 = \frac{j \cdot 2\pi}{4} \cdot 1 = \frac{j\pi}{2}$

$Y_{-2} = \frac{j(-2\pi)}{4} \cdot 1 = -\frac{j\pi}{2}$

$Y_k = 0$ o/w

10 **Problem 5** (15 pts) Consider the following periodic signals $x(t)$ and $y(t)$





4 (a) (5 pts) Determine the complex exponential Fourier series of $x(t)$, i.e., find the coefficients X_k that satisfy

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 kt}$$

4 (b) (5 pts) Determine the trigonometric Fourier series of $x(t)$, i.e., find the coefficients a_k and b_k that satisfy

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) - b_k \sin(k\omega_0 t)$$

2 (c) (5 pts) Using part (a), determine the complex exponential Fourier series of $y(t)$, i.e., find the coefficients Y_k that satisfy

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j\omega_0 kt}$$

a) $X_0 = \frac{A}{2}$

$X_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$

$X_k = \frac{1}{T_0}\int_0^{T_0/2} A e^{-jk\omega_0 t} dt$

$X_k = \frac{A}{T_0}\int_0^{T_0/2} e^{-jk\omega_0 t} dt$

$X_k = \frac{A}{T_0}\left(\frac{1}{-jk\omega_0}\right) e^{-jk\omega_0 t}\Big|_0^{T_0/2}$ $\qquad \omega_0 = \frac{2\pi}{T_0} \quad \therefore \frac{1}{T_0}\cdot\frac{1}{\omega_0} = \frac{1}{2\pi}$

$X_k = \frac{A}{2\pi}\left(\frac{j}{k}\right)[-1-1]$

$X_k = -\frac{A}{\pi}\left(\frac{j}{k}\right)$ odd

b) $a_k = Re\{X_k\} = 0$

$b_k = Im\{X_k\} = -\frac{A}{\pi k}$ odd

$x(t) = \frac{A}{2} + 2\sum_{k=1}^{\infty} \frac{A}{\pi k} \sin(k\omega_0 t)$

c) $y(t) = x(t) - x(t - \frac{T_0}{2})$ $\qquad$ time-shifting: $x(t-\varphi) \xrightarrow{F} e^{-j\varphi\Omega_0} X_k$

for $k \neq 0$ $Y_k = X_k - \cancel{X_k} e^{-j\pi} = \boxed{2X_k}$

$\boxed{Y_0 = 0}$

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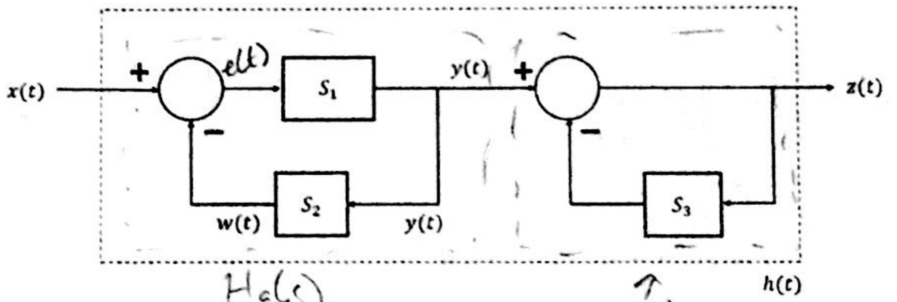
Problem 6 (15 pts) Consider the following LTI system with input $x(t)$, output $z(t)$, and impulse response function (IRF) $h(t)$. Systems S_1, S_2 and S_3 are LTI systems as well with the following characteristics

1. IRF of S_1 is $h_1(t) = e^{-2t}u(t)$.
2. Input-output relation for S_2 is $w(t) = \frac{d}{dt}y(t)$
3. IRF of S_3 is $h_3(t) = e^{-t}u(t)$.

(Assume all the signals at $t = 0$ are zero, i.e., all initial conditions are zero).



$H_a(s)$ $H_b(s)$

5 (a) (8 pts) Compute the IRF of entire system $h(t)$ and its Laplace transform $H(s)$.

3 (b) (7 pts) Find the output $z(t)$ if the following input is applied to the system:

$$x(t) = e^{-2t}\cos(8t - 24)u(t - 3) \quad (1)$$

a) $z(t) = y(t) - h_3(t) * y(t)$ $\quad Z(s) = Y(s) - H_3(s)Y(s)$

$H_b(s) = \frac{Z(s)}{Y(s)} = 1 - H_3(s) = 1 - \frac{1}{s+1} = \frac{s}{s+1}$

$e(t) = x(t) - w(t)$ $\quad w(t) = \frac{d}{dt}y(t)$ $\quad y(t) = e(t) * h_1(t)$

$W(s) = sY(s) \cancel{- y(0)}$ $\quad Y(s) = E(s)H_1(s)$

$H_1(s) = \frac{1}{s+2}$ $\quad Y(s) = (X(s) - sY(s))H_1(s)$

$H_a(s) = \frac{Y(s)}{X(s)} = H_1(s) - \frac{sY(s)H_1(s)}{X(s)}$

$(1 + sH_1(s))H_a(s) = H_1(s)$

$H_a(s) = \frac{H_1(s)}{1 + sH_1(s)} = \frac{\frac{1}{s+2}}{1 + \frac{s}{s+2}} = \frac{1}{(s+2)\left(\frac{s+2+s}{s+2}\right)} = \frac{1}{2s+2}$

$$H(s) = H_a(s)H_b(s) = \frac{1}{2(s+1)}\left(\frac{s}{s+1}\right) = \boxed{\frac{s}{2(s+1)^2}}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$H(s) = \frac{1}{2}\left(\frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\right) = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$h(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-t}r(t)$$

b) $x(t) = e^{-2t}\cos(8(t-3))u(t-3)$

$$X(s) = e^{-3s}\frac{s+2}{(s+2)^2+64}$$

$$Z(s) = X(s)H(s) = e^{-3s}\left(\frac{s+2}{s^2+4s+68}\right)\left(\frac{s}{2(s+1)^2}\right)$$

$$Z(s) = \frac{1}{2}e^{-3s}\left(\frac{s+2}{s^2+4s+68}\right)\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right) = \frac{1}{2}e^{-3s}\left(\frac{1}{s^2+4s+68} + \frac{s+1}{s^2+4s+68}\right)\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$Z(s) = \frac{1}{2}e^{-3s}\left(\cancel{\frac{1}{s^2+4s+68}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)} + \frac{1}{s^2+4s+68}\left(1 - \frac{1}{s+1}\right)\right)$$

$$z(t) = h(t) * x(t)$$

$$z(t) = \frac{1}{2}\int_{0^-}^{\infty}\left(e^{-\tau}u(t) - e^{-\tau}r(\tau)\right)\left(e^{-2(t-\tau)}\cos(8(t-\tau-3))\,u(t-\tau-3)\right)d\tau$$

$$z(t) = \frac{1}{2}u(t)\int_{0^-}^{t-3}\left(e^{-\tau} - \tau e^{-\tau}\right)\left(e^{-2(t-\tau)}\left(\frac{1}{2}\left(e^{8j(t-\tau-3)} + e^{-8j(t-\tau-3)}\right)\right)\right)d\tau$$

$$z(t) = \frac{1}{4}e^{-2t}u(t)\int_{0^-}^{t-3}\left(e^{\tau} - \tau e^{\tau}\right)\left(e^{8j(t-\tau-3)} + e^{-8j(t-\tau-3)}\right)d\tau$$

$$z(t) = \frac{1}{4}e^{-2t}u(t)\left(e^{8j(t-3)}\int_{0^-}^{t-3} e^{\tau(1-8j)} - \tau e^{\tau(1-8j)}\,d\tau + e^{-8j(t-3)}\int_{0^-}^{t-3} e^{\tau(1+8j)} - \tau e^{\tau(1+8j)}\,d\tau\right.$$