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PART I

Time-Domain Analysis

(Do Not Use Laplace Transforms for Questions 1 and 2)!

Question 1 (15 pt)

(i) (10 pt) The IPOP relation of a L system S is:

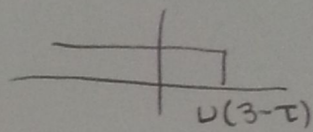
$$y(t) = \int_{-\infty}^{\infty} t(t-\tau)u(t-\tau)x(\tau)d\tau, \ t \in (-\infty, \infty),$$

where

$$x(t) \to [S] \to y(t).$$

Write down the IRF $h(t,\tau)$ of S. Then compute its output $y(t)$ given that its input $x(t)$ is

$$x(t) = u(t)u(3-t), \ t \in (-\infty, \infty).$$



(ii) (5 pt) S is: TV? TI? C? NC?

(i) $y(t) = \int_{-\infty}^{\infty} \underbrace{t(t-\tau)\cdot u(t-\tau)}\, x(\tau)\, d\tau$

IRF: $\boxed{h(t) = t(t-\tau)\cdot u(t-\tau)}$

$y(t) = \int_{-\infty}^{\infty} t(t-\tau)\cdot u(t-\tau)\cdot u(\tau)\cdot u(3-\tau)\, d\tau$

- $t<0$: $y(t)=0$ b/c $y(t) = \int_{-\infty}^{\infty} t(t-\tau)\cdot \underbrace{u(\tau)}_{0 \text{ for } t<0} d\tau$

- $0 \le t \le 3$: $y(t) = \int_0^t t\cdot(t-\tau)\,d\tau = t^2\int_0^t d\tau - t\int_0^t \tau\, d\tau$

  $y(t) = t^3 - t\left(\frac{1}{2}t^2\right) = \boxed{\frac{1}{2}t^3}$

- $t>3$: $y(t) = \int_0^3 t(t-\tau)\,d\tau = t^2\int_0^3 d\tau - t\int_0^3 \tau\, d\tau$

  $= 3t^2 - t\left(\frac{1}{2}\tau^2\Big|_0^3\right) = \boxed{3t^2 - \frac{9}{2}t}$

$$\boxed{y(t) = \begin{cases} 0 & : t<0 \\ \frac{1}{2}t^3 & : 0 \le t \le 3 \\ 3t^2 - \frac{9}{2}t & : t \ge 3 \end{cases}}$$

OR $y(t) = u(t)\cdot\left[\frac{1}{2}t^3\cdot u(3-t) + \left[3t^2 - \frac{9}{2}t\right]\cdot u(t-3)\right]$

(ii) S : $\boxed{TV}$ because $h(t,\tau) \neq h(t-\tau, 0)$

$h(t,\tau) = t(t-\tau)\cdot u(t-\tau)$

$h(t-\tau,0) = (t-\tau)\cdot(t-\tau)\cdot u(t-\tau)$

$h(t,\tau) \neq h(t-\tau, 0)$

$\boxed{\text{Causal}}$ because the system does not depend on future values of t.

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Question 2 (15 pt)
Find the impulse response function $h(t, \tau)$ of the cascaded systems $S_1 S_2$ with the following input-output descriptions:

$$x(t) \to [S_1] \to z(t) \to [S_2] \to y(t),$$

where

$$z(t) = e^{-t} x(t) u(t),$$

and

$$y(t) = \int_0^t e^{-(t-\sigma)} z(\sigma) u(\sigma) d\sigma, \ t \geq 0.$$

let $x(t) = \delta(t)$

$\hookrightarrow z(t) = e^{-t} \delta(t) u(t)$

$$y(t) = \int_0^t e^{-(t-\sigma)} e^{-\sigma} \delta(\sigma) u(\sigma) d\sigma$$

$$y(t) = \int_{-\infty}^{\infty} e^{-t} e^{\sigma} e^{-\sigma} \delta(\sigma) u(\sigma) u(t-\sigma) d\sigma$$

$$y(t) = \int_{-\infty}^{\infty} e^{-t} \cdot u(t-\sigma) \cdot \delta(\sigma) \cdot u(\sigma) d\sigma \qquad \tau = t - \sigma$$

$h_{12}(t,\tau)$: $\boxed{h(t,\tau) = e^{-t} \cdot u(t-\tau)}$ $u(t)$.

$$y(t) = e^{-t} \int_{-\infty}^{\infty} u(\tau) \delta(t-\tau) \cdot u(t-\tau) d\tau$$

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PART II

S-Domain Analysis

(Use Laplace Transforms for Questions 3 and 4)!

Question 3 (15 pt)

The figure below is the "poles-zeros" plot of a system function $H(s)$:

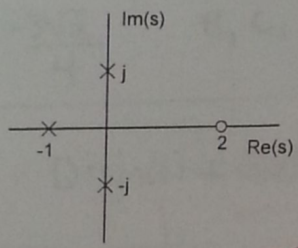
(i) (5 pt) Find $H(s)$ given $H(0) = \sqrt{2}$.

(ii) (10 pt) Find the output of the system when $tu(t-1)$ is applied, knowing that the system was at rest.

(i)

Poles: $s = +j$, $s = -j$, $s = -1$

zero: $s = 2$



$$H(s) = \frac{A(s-2)}{(s-j)(s+j)(s+1)}$$

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$$\boxed{H(s) = \frac{-\frac{\sqrt{2}}{2}(s-2)}{(s-j)(s+j)(s+1)}}$$

$$H(0) = \frac{-2A}{(-j)(j)(1)} = \frac{-2}{1}A = \sqrt{2} \qquad A = \frac{-\sqrt{2}}{2}$$

(ii) $x(t) = t\cdot \underbrace{u(t-1)}_{f(t)}$ $\qquad X(s) = \frac{-d}{ds}[F(s)]$ $\qquad$ where $F(s) = \frac{e^{-s}}{s}$

$$X(s) = \frac{-d}{ds}\left[\frac{e^{-s}}{s}\right] = -\left[\frac{-e^{-s}\cdot s - e^{-s}}{s^2}\right] = \frac{1+se^{-s}}{s^2} = \frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$Y(s) = X(s)\cdot H(s) = \frac{-\frac{\sqrt{2}}{2}(s-2)\cdot(1+e^{-s})}{s^2(s-j)(s+j)(s+1)} = (1+e^{-s})\left[\cancel{\frac{As+B}{s^2} + \frac{C}{s-j} + \frac{D}{s+j} + \frac{E}{s+1}}\right]$$

$$\cancel{-\frac{\sqrt{2}}{2}s + \sqrt{2} = (As+B)(s^2+1)(s+1)}$$

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$$H(s) = \frac{-\frac{\sqrt{2}}{2}(s-2)}{(s-j)(s+j)(s+1)} \qquad X(s) = \frac{1+se^{-s}}{s^2}$$

$$Y(s) = (1+se^{-s})\left[\frac{-\frac{\sqrt{2}}{2}(s-2)}{s^2(s-j)(s+j)(s+1)}\right] = (1+se^{-s})\left[\frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} + \frac{E}{s+1}\right]$$

$$A = \underline{\underline{\sqrt{2}}} \qquad E = \frac{-\frac{3\sqrt{2}}{2}}{(-1-j)(-1+j)} = \frac{-\frac{3\sqrt{2}}{2}}{2} = \underline{\underline{\frac{-3\sqrt{2}}{4}}}$$

$$-\frac{\sqrt{2}}{2}s + \sqrt{2} = (\sqrt{2}s+B)(s^2+1)(s+1) + (Cs+D)(s^2)(s+1) - \frac{3\sqrt{2}}{4}(s^2)(s^2+1)$$

$$A = \sqrt{2} \qquad E = \frac{-3\sqrt{2}}{4} \qquad B, C, D = \ldots$$

$$\boxed{y(t) = [\sqrt{2} + Bt + C\cdot\cos(t) + D\cdot\sin(t) + Ee^{-t}]\cdot u(t) + [\sqrt{2} + Bt + C\cdot\cos t + D\sin t + Ee^{-t}]\cdot u(t-1)}$$

? $(t-1)$

Question 4 (8 pt)
Given $x(t) = \sqrt{2}e^{-t}u(t)$ and $y(t) = \sqrt{2}e^{-t}u(t)(1-2t)$, find the impulse response function $h(t)$ of the linear time-invariant system which admits the given input-output pair.

$$x(t) = \sqrt{2}\, e^{-t} \cdot u(t) \rightarrow X(s) = \frac{\sqrt{2}}{s+1}$$

$$y(t) = \sqrt{2}\, e^{-t} \cdot u(t)(1-2t) = \sqrt{2}\, e^{-t} \cdot u(t) - 2\sqrt{2}\, t\, e^{-t} \cdot u(t)$$

$$Y(s) = \frac{\sqrt{2}}{s+1} - \frac{2\sqrt{2}}{(s+1)^2} = \frac{\sqrt{2}(s+1) - 2\sqrt{2}}{(s+1)^2}$$

$$Y(s) = \frac{\sqrt{2}(s-1)}{(s+1)^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\cancel{\sqrt{2}}(s-1)}{(s+1)^{\cancel{2}}} \cdot \left(\frac{\cancel{s+1}}{\cancel{\sqrt{2}}}\right) = \frac{s-1}{s+1}$$

$$\begin{array}{r} 1 \\ s+1 \,\overline{)\, s-1} \\ -(s+1) \\ \hline -2 \end{array}$$

$$\boxed{H(s) = 1 - \frac{2}{s+1}}$$

$$\boxed{h(t) = \delta(t) - 2e^{-t} \cdot u(t)} \quad 8$$

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PART III

Time-Domain and/or S-Domain Analysis

(Use whatever methods you are most comfortable with)!

Question 5 (17 pt)
The system function $H(s)$ of a linear time-invariant systems S is

$$H(s) = \frac{1}{s^2 + s + 1}.$$

(i) (2 pt) If $x(t)$ and $y(t)$ are the input and output of S, respectively, find the differential equation relating $x(t)$ and $y(t)$. Assume the system is at rest.

(ii) (10 pt) Find $y(t)$ when $x(t) = \sin(2(t-1))u(t-1)$.

(iii) (5 pt) Find the step response $g(t)$ of S by direct calculation in the time domain and by Laplace transform.

(i) $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+s+1}$ $\quad Y(s)\cdot[s^2+s+1] = X(s)$

$$\boxed{\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)}$$

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(ii) $Y(s) = \frac{X(s)}{s^2+s+1}$ $\quad x(t) = \sin[2(t-1)]\cdot u(t-1)$

$X(s) = e^{-s}\cdot\frac{2}{s^2+4}$

2

$$Y(s) = \frac{2e^{-s}}{(s^2+s+1)(s^2+4)} = e^{-s}\left[\frac{As+B}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{Cs+D}{s^2+4}\right]$$

$s^2+4=0$
$s^2=-4$
$s=\pm j2$

$$2 = (As+B)\cdot(s^2+4) + (Cs+D)(s^2+s+1)$$

s^3: $0 = A + C$ $\quad A = -C$

s^2: $0 = B + C + D$ $\longrightarrow$ $0 = \frac{1}{2} - \frac{D}{4} + C + D$

s: $0 = 4A + C + D \rightarrow 0 = -3C + D$ $\quad D = 3C$

2: $2 = 4B + D$ $\quad B = \frac{2-D}{4}$ $\quad -\frac{1}{2} = \frac{3}{4}D + C$

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$-\frac{1}{2} = \frac{3}{4}(3C) + C$

$A = \frac{2}{13}$ $\quad D = \frac{-6}{13}$ $\quad C = \frac{-4}{26} = \frac{-2}{13} \longleftarrow$ $\quad \frac{13}{4}C = -\frac{1}{2}$

$$B = \frac{\frac{26}{13} - (\frac{-6}{13})}{4} = \frac{8}{13}$$

$$Y(s) = \left[\left(\frac{\frac{2}{13}s + \frac{8}{13}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right) - 2\left(\frac{\frac{2}{13}s + \frac{6}{13}}{\cancel{2}s^2+4}\right)\right] e^{-s}$$

$\frac{6}{13}y = 2$ $\quad y = \frac{26}{6}$ $\quad \frac{8}{13}x = \frac{\sqrt{3}}{2}$ $\quad x = \frac{13\sqrt{3}}{16}$

$e^{-\frac{1}{2}t} \cdot f(t) = F(s+\frac{1}{2})$

$$\boxed{y(t) = \left[e^{-\frac{1}{2}t}\left[\frac{2}{13}\cdot\cos\left(\frac{\sqrt{3}}{2}\right)t + \frac{16}{13\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] - \left[\frac{2}{3}\cos(2t) + \frac{3}{13}\sin(2t)\right]\right]\cdot u(t-1)}$$

-1

(iii) step

$$G(s) = s\cdot H(s) = \frac{s}{s^2+s+1} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\boxed{g(t) = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)\cdot u(t) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\cdot u(t)}$$

$$H(s) = \frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$h(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)\cdot u(t)$$

time domain

$$\boxed{\begin{aligned} g(t) &= \int_{-\infty}^{\infty} h(t-\tau)\cdot u(\tau)\,d\tau \\ &= \int_0^t \frac{2}{\sqrt{3}}e^{-\frac{1}{2}\tau}\cdot\sin\left(\frac{\sqrt{3}}{2}(t-\tau)\right)d\tau \end{aligned}}$$

$\int u\,dv = uv - \int v\,du$

$\frac{d}{ds}(s+1)^{-1} = -(s+1)^{-2}$ $\qquad f(t-d)\cdot u(t-d) = e^{-sd}\cdot F(s)$

$\frac{d}{ds} -(s+1)^{-2} = 2(s+1)^{-3}$ $\qquad e^{-\alpha t} f(t) = F(s+\alpha)$

Question 6 (10 pt)

Compute

$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)u(t-\sigma)x(\sigma)d\sigma,$$

for

(i) (5 pt) $h(t) = t^2u(t)$, $x(t) = e^{-t}u(t-5)$.

(ii) (5 pt) $h(t) = \sin(\omega t)$, $x(t) = tu(t)$.

(i) $y(t) = \int_{-\infty}^{\infty} (t-\sigma)^2 u(t-\sigma)\cdot e^{-\sigma}\cdot u(\sigma-5)\,d\sigma$

- $H(s) = \frac{2}{s^3}$ $\quad F(s) = \frac{e^{-5s}}{s}$ $\quad X(s) = F(s+1)$
- $X(s) = \frac{e^{-5(s+1)}}{s+1}$ $\quad Y(s) = X(s)\cdot H(s)$

$Y(s) = \frac{2e^{-5(s+1)}}{s^3(s+1)} = e^{-5(s+1)}\left[\frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+1}\right]$

$A = \frac{1}{1} = \underline{1}$ $\quad B = \frac{d}{ds}\left[\frac{1}{s+1}\right]\Big|_{s=0} = \frac{-1}{(s+1)^2}\Big|_{s=0} = \underline{-1}$

$C = \frac{d^2}{ds^2}\left[\frac{1}{s+1}\right]\Big|_{s=0} = \frac{2}{(s+1)^3}\Big|_{s=0} = \underline{2}$ $\quad D = \lim_{s\to -1} \frac{1}{s^3} = \frac{1}{(-1)^3} = -1$

$Y(s) = e^{-5(s+1)}\left[\frac{1}{s^3} - \frac{1}{s^2} + \frac{2}{s} - \frac{1}{s+1}\right]$

$= e^{-5s}\cdot e^{-5}\left[\frac{1}{s^3} - \frac{1}{s^2} + \frac{2}{s} - \frac{1}{s+1}\right]$

$y(t) = e^{-5}\cdot u(t-5)\left[\frac{1}{2}t^2 - t + 2 - e^{-t}\right]$

$$\boxed{y(t) = \left[\frac{1}{2}t^2 - t + 2 - e^{-t}\right]\cdot e^{-5}\cdot u(t-5)}$$

(ii) $y(t) = \int_{-\infty}^{\infty} h(t-\sigma)\cdot u(t-\sigma)\, x(\sigma)\, d\sigma$

$h(t) = \sin(\omega t) \qquad x(t) = t \cdot u(t)$

$H(s) = \dfrac{\omega}{s^2+\omega^2} \qquad X(s) = \dfrac{1}{s^2}$

$Y(s) = \dfrac{\omega}{s^2(s^2+\omega^2)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{Cs+D}{s^2+\omega^2}$

$A = \dfrac{\omega}{\omega^2} = \dfrac{1}{\omega} \qquad B = \dfrac{d}{ds}\left[\dfrac{\omega}{s^2+\omega^2}\right]\Bigg|_{s=0} = \dfrac{0-\omega\cdot 2s}{(s^2+\omega^2)^2}\Bigg|_{s=0} = 0$

$\omega = \frac{1}{\omega}(s^2+\omega^2) + (Cs+D)\,s^2$

$0 = \frac{1}{\omega} + D \qquad D = \frac{-1}{\omega}$

$\omega = \frac{1}{\omega}\cdot\omega$ ✓

$0 = Cs^3 \rightarrow C = 0$

$Y(s) = \dfrac{\frac{1}{\omega}}{s^2} - \dfrac{\frac{1}{\omega}\cdot\frac{\omega^2}{\omega^2}}{s^2+\omega^2}$

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$$\boxed{y(t) = \frac{1}{\omega}t\cdot u(t) - \frac{1}{\omega^2}\sin(\omega t)\cdot u(t)}$$

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