UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

ECE102: SYSTEMS & SIGNALS

Midterm Examination
February 25, 2020
Duration: 1 hr 50 mins.

INSTRUCTIONS:

- The exam has 5 problems and 13 pages.
- The exam are closed-book.
- Two cheat sheets of A4 size is allowed.
- Calculator is NOT allowed.
- Write your discussion session in the top-right corner. ↗↗

Your name:____________________________________

Student ID:____________________________________

Table 1: Score Table

Problem	a	b	c	d	e	Score
1	10	10				20
2	5	5	10			20
3	5	10	5			20
4	15					15
5	5	10	10			25
Total						100

Problem 1 (20 pts)

Find the Laplace transforms and ROC of the following signals

(a) (10 pts) $x_1(t) = (t-2)e^{-3t+6}\cos(4t-8)u(t-2)$

(b) (10 pts) $x_2(t) = \int_0^t \sin^2(t-\tau)\cos^2(\tau)d\tau$

Solution:

(a)

$$\mathcal{L}\left\{e^{-3t}\cos(4t)u(t)\right\} = \frac{s+3}{(s+3)^2+4^2}, \mathcal{R}\{s\} > -3$$

$$\begin{aligned}\mathcal{L}\left\{te^{-3t}\cos(4t)u(t)\right\} &= -\frac{1(s^2+6s+25)-(2s+6)(s+3)}{(s^2+6s+25)^2} \\ &= \frac{s^2+6s-7}{(s^2+6s+25)^2}\end{aligned}$$

$$\mathcal{L}\left\{(t-2)e^{-3(t-2)}\cos(4(t-2))u(t-2)\right\} = \frac{e^{-2s}(s^2+6s-7)}{(s^2+6s+25)^2}, \mathcal{R}\{s\} > -3$$

(b)

$$\begin{aligned}&\int_0^t \sin^2(t-\tau)\cos^2(\tau)d\tau \\ =&\int_{-\infty}^{\infty} \sin^2(t-\tau)u(t-\tau)\cos^2(\tau)u(\tau)d\tau \\ =&\int_{-\infty}^{\infty} \frac{1-\cos(2(t-\tau))}{2}\frac{1+\cos(2\tau)}{2}d\tau \\ =&\frac{1-\cos(2t)}{2}u(t) * \frac{1+\cos(2t)}{2}u(t)\end{aligned}$$

$$\mathcal{L}\left\{\frac{1-\cos(2t)}{2}u(t)\right\} = \frac{1}{2}(\frac{1}{s} - \frac{s}{s^2+2^2}), \mathcal{R}\{s\} > 0$$

$$
\begin{aligned}
&\mathcal{L}\left\{\int_0^t \sin^2(t-\tau)\cos^2(\tau)d\tau\right\} \\
=&\frac{1}{2}(\frac{1}{s}-\frac{s}{s^2+2^2})\times\frac{1}{2}(\frac{1}{s}+\frac{s}{s^2+2^2}) \\
=&\frac{2s^2+4}{s^2(s^2+4)^2}, \mathcal{R}\{s\}>0
\end{aligned}
$$

Problem 2 (20 pts)

The input-output relationship for an LTI system is given by the following differential equation

$$3\frac{d^2y(t)}{dt^2}+19\frac{dy(t)}{dt}+20y(t)=2\frac{dx(t)}{dt}-x(t), t\geq 0$$

with initial conditions $y'(0)=y(0)=0, x(0)=0$.

(a) (5 pts) Find the transfer function $H(s)$, and determine whether the system is BIBO stable or not.

(b) (5 pts) Find impulse response function $h(t)$.

(c) (10 pts) Find the output $y(t)$ if the input is $x(t)=e^{\frac{1}{2}(t-3)}u(t-3)$.

Solution:

(a) Take LT of both sides to get

$$3s^2Y(s)+19Y(s)+20Y(s)=2sX(s)-X(s),$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2s-1}{3s^2+19s+20}=\frac{2s-1}{(3s+4)(s+5)}$$

The poles are $s=-4/3, -5$, so the system is BIBO stable.

(b)

$$
\begin{aligned}
H(s)&=\frac{2s-1}{3s^2+19s+20} \\
&=\frac{-1}{3s+4}+\frac{1}{s+5}
\end{aligned}
$$

Take inverse LT to get

$$h(t)=\frac{-1}{3}e^{-\frac{4}{3}}u(t)+e^{-5t}u(t)$$

(c) Let $x_1(t) = e^{\frac{1}{2}t}u(t)$ then $X_1(s) = \frac{1}{s-1/2}$. Since $x(t) = x_1(t-3)$, we have using time shift property of LT

$$X(s) = \frac{2e^{-3s}}{2s-1}$$

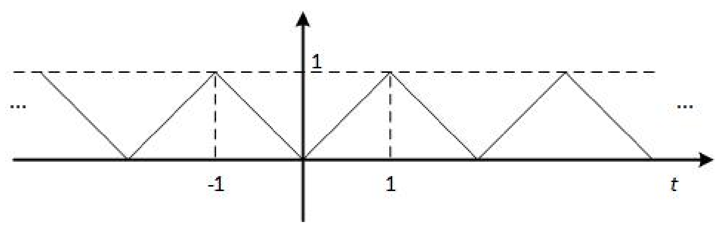
$$Y(s) = H(s)X(s) = \frac{2s-1}{3s^2+19s+20}\frac{2e^{-3s}}{2s-1} = \frac{2e^{-3s}}{3s^2+19s+20}$$

Let $y_1(t) = \mathcal{L}^{-1}\frac{2}{3s^2+19s+20} = \mathcal{L}^{-1}\frac{2}{11}(\frac{3}{3s+4} - \frac{1}{s+5}) = \frac{2}{11}e^{-\frac{4}{3}t}u(t) - \frac{2}{11}e^{-5t}u(t)$. Then, using the time-shift property again we get

$$y(t) = y_1(t-3) = \frac{2}{11}e^{-\frac{4}{3}(t-3)}u(t-3) - \frac{2}{11}e^{-5(t-3)}u(t-3)$$

Problem 3 (20 pts)

Consider the following periodic signal $x(t)$. For $-1 < t < 1$, the mathe-



matical expression of $x(t)$ is

$$x(t) = \begin{cases} -t, & -1 < t < 0 \\ t, & 0 \le t < 1 \end{cases}$$

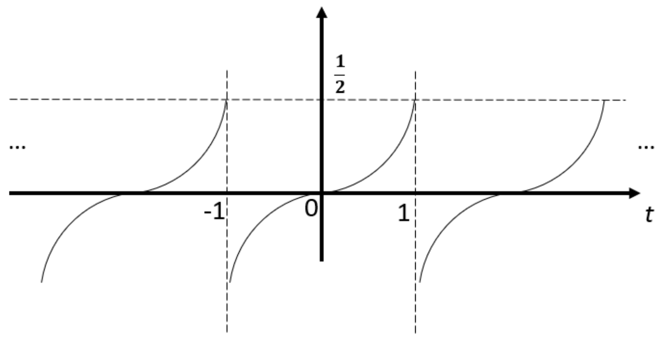
(a) (5 pts) Find the Fourier series coefficients X_k.

(b) (10 pts) Show that if the Fourier series coefficients of a periodic signal $a(t)$ are A_k and $b(t) = \frac{da(t)}{dt}$, then the Fourier series coefficients of $b(t)$, B_k, are

$$B_k = jk\omega \times A_k, \quad \omega = \frac{2\pi}{T}$$

(c) (5 pts) Use the property in (b) to find the Fourier series coefficients of the periodic signal $y(t)$, where the period of $y(t)$ is 2. For $-1 < t < 1$, the mathematical expression of $y(t)$ is

$$y(t) = \begin{cases} -\frac{1}{2}t^2, & -1 < t < 0 \\ \frac{1}{2}t^2, & 0 \le t < 1 \end{cases}$$



Solution:
$T = 2, \omega = \frac{2\pi}{T} = \pi$
a)

The signal can be expressed in another way. Consider time interval $0 < t < 2$. In this interval, we define $x_1(t)$ to be

$$x_1(t) = r(t) - 2r(t-1) + r(t-2), \quad 0 < t < 2$$

Then $X_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega t}dt = \frac{1}{T}\int_0^2 x_1(t)e^{-jk\omega t}dt = \frac{1}{T}X_1(s = jk\omega)$

By using the Laplace transform table, $X_1(s) = \frac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$. So,

$$\begin{aligned} X_k &= \frac{1}{2(jk\pi)^2}(1 - 2e^{-jk\pi} + e^{-jk2\pi}) \\ &= \frac{2(-1)^k - 2}{2k^2\pi^2} \\ &= \frac{(-1)^k - 1}{k^2\pi^2}, k \neq 0 \end{aligned}$$

X_0 Here that $e^{-jk2\pi} = 1$ and $e^{-jk\pi} = (-1)^k$ is used. For $k = 0$, $X_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2}$
b)

Let $a(t) = \sum_{-\infty}^{\infty} A_k e^{jk\omega t}$. Since $b(t) = \frac{da(t)}{dt}$, we have

$$
\begin{aligned}
b(t) &= \frac{da(t)}{dt} \\
&= \sum_{-\infty}^{\infty} A_k \frac{de^{jk\omega t}}{dt} \\
&= \sum_{-\infty}^{\infty} jk\omega A_k e^{jk\omega t}
\end{aligned}
$$

The last equation is the Fourier series expansion for $b(t)$. So $B_k = jk\omega \times A_k$.
c)

Here, we can find that $x(t) = \frac{dy(t)}{dt}$. By using the property we showed in (b) we have

$$
\begin{aligned}
X_k &= jk\omega Y_k \\
\Rightarrow Y_k &= \frac{X_k}{jk\omega}, k \neq 0 \\
\Rightarrow Y_k &= \frac{(-1)^k - 1}{jk^3\pi^3}, k \neq 0
\end{aligned}
$$

For $k = 0$, $Y_0 = \frac{1}{T}\int_{-1}^{1} y(t)dt = 0$ since $y(t)$ is an odd function.

Problem 4 (15 pts)

The following information is given for a real periodic signal $x(t)$ with period $T_0 = 2\pi$, where X_k is its Fourier series coefficients.

- $x(t)$ is a even function
- $X_k = 0$ for $|k| \geq 3$
- $\int_{-\pi}^{\pi} x(t)dt = 0$
- The value of the signal at time instant $t = 0$ is 2, i.e., $x(0) = 2$
- The power of $x(t)$ is $\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = 2$

Find its Fourier series coefficients X_k and determine the time domain signal $x(t)$.

Solution:

The fundamental frequency is $\Omega_0 = \frac{2\pi}{T_0} = 1$. Due to the fact $X_k = 0, |k| \geq 3$, we can determine $x(t)$ by

$$x(t) = X_0 + X_1 e^{jt} + X_{-1}e^{-jt} + X_2 e^{j2t} + X_{-2}e^{-j2t}$$

For the DC component, we use

$$X_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)dt = 0$$

Since the signal is real and even, we have $X_k = X_{-k} = X^*_{-k}$. So all the Fourier series coefficients are real and $X_1 = X_{-1}, X_2 = X_{-2}$. Let $X_1 = X_{-1} = a$ and $X_2 = X_{-2} = b$. Using $x(0) = 2$, we have

$$\begin{aligned} &x(0) = 0 + a + a + b + b = 2a + 2b = 2 \\ &\Rightarrow a + b = 1 \end{aligned}$$

Due to the power is 2, the Parseval's relation gives

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} |X_k|^2 = |b|^2 + |a|^2 + |a|^2 + |b|^2 = 2|a|^2 + 2|b|^2 = 2a^2 + 2b^2 = 2 \\ &\Rightarrow a^2 + b^2 = 1 \end{aligned}$$

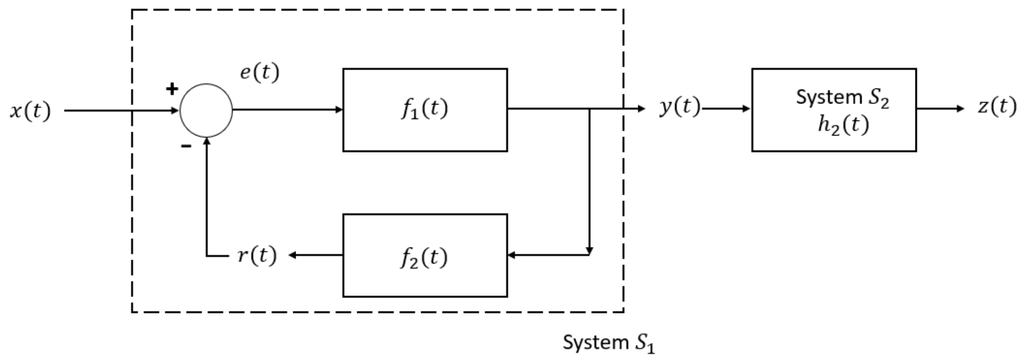
Substituting $b = 1 - a$ into $a^2 + b^2 = 1$, we get $(a, b) = (0, 1)$ or $(a, b) = (1, 0)$. Since the fundamental period period for $x(t)$ is $T_0 = 2\pi$, we should choose $(a, b) = (1, 0)$ and $x(t) = e^{jt} + e^{-jt} = 2\cos(t)$

Problem 5 (25 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.

(a) (5 pts) Find the transfer function $H_1(s)$ of system S_1. Express $H_1(s)$ in terms of $F_1(s)$ and $F_2(s)$, where $F_1(s), F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$.

Hint: $e(t) = x(t) - r(t)$



(b) (10 pts) Find the transfer function $H_{12}(s)$ of the cascaded system S_{12}, where

$$f_1(t) = t^2 e^{-4t} u(t)$$
$$f_2(t) = u(t-1) - u(t-2)$$
$$h_2(t) = e^{-2t} \cos^2(3t) u(t)$$

(c) (10 pts) Let $z(t)$ be the steady-state response due to the periodic input signal

$$x(t) = 1 + 2\sin(t) + 3\cos(2t)$$

Find exponential Fourier series coefficients of $z(t)$, Z_k.

Solution:

(a) We have $E(s) = X(s) - R(s)$ and $Y(s) = F_1(s)E(s) = F_1(s)(X(s) - R(s)) = F_1(s)(X(s) - F_2(s)Y(s))$. Therefore

$$Y(s) = F_1(s)[X(s) - F_1(s)F_2(s)Y(s)]$$
$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{F_1(s)}{1 + F_1(s)F_2(s)}$$

(b) The transfer function $H_{12}(s) = H_1(s)H_2(s)$

$$\begin{aligned}
h_2(t) &= e^{-2t}\cos^2(3t)u(t) \\
&= e^{-2t}\frac{1+\cos(6t)}{2}u(t) \\
H_2(s) &= \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+6^2}\right] \\
F_1(s) &= \frac{2}{(s+4)^3} \\
F_2(s) &= \frac{1}{s}(e^{-s} - e^{-2s}) \\
H_{12}(s) &= \frac{\frac{2}{(s+4)^3} \times \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{s^2+4s+40}\right]}{1 + \frac{2}{(s+4)^3} \times \frac{e^{-s}-e^{-2s}}{s}}
\end{aligned}$$

(c) $\omega_0 = 2\pi/2\pi = 1$.

First, find X_k. $x(t)$ can be written as follows using Euler's identity

$$x(t) = 1 + \frac{2}{2j}e^{jt} + \frac{-2}{2j}e^{-jt} + \frac{3}{2}e^{j2t} + \frac{3}{2}e^{-j2t}$$

Comparing with $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkt}$, we get the following coefficients:

$$\begin{aligned}
&X_0 = 1; \\
&X_{-1} = \frac{-1}{j}, \quad X_1 = \frac{1}{j} \\
&X_{-2} = \frac{3}{2}, \quad X_2 = \frac{3}{2}
\end{aligned}$$

Using $Z_k = H(jk\omega_0)X_k$, we get $Z_k = H(jk)X_k, \quad k = 0, \pm1, \pm2$