UCLA DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ECE 102: SYSTEMS & SIGNALS

Midterm Examination
February 2, 2021
Duration: 1 hr 50 min. (+15 min. for Gradescope submission)

INSTRUCTIONS:

- The exam has 6 problems and 18 pages.
- The exam is open-book and open-notes.
- Calculator/MATLAB allowed.
- Show all of your work! No credit given for answers without math steps shown and/or an explanation.
- NO LATE SUBMISSIONS ALLOWED ON GRADESCOPE.

Your name: Anna Anderson

Student ID: 105296576

Table 1: Score Table

Problem	a	b	c	d	e	Score
1	8	7				15
2	3	3	2	4		12
3	2	2	4	3	4	15
4	6	6	4	2		18
5	3	5				8
6	4	8				12
Total						80

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t)$, $a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t)$, $a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t)$, $a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t)$, $a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t)$, $\beta g(t)$	$\alpha F(s)$, $\beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)$ $\alpha \neq 0$	$\frac{1}{\vert\alpha\vert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

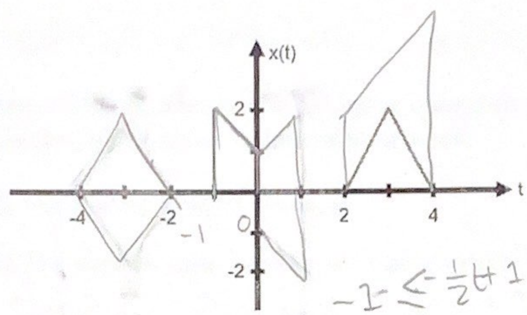
Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Problem 1 (15 pts)

Consider signal $x(t)$ depicted in the figure below



$2 \le -\frac{1}{2}t+1 \le 4$

≤ 3

$-2 \ge t \ge -6$

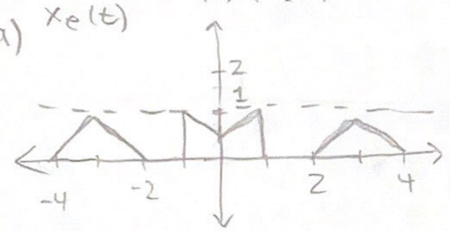
$-1 \le -\frac{1}{2}t+1 \le 0$

$-2 \quad 4 \ge t \ge 2$

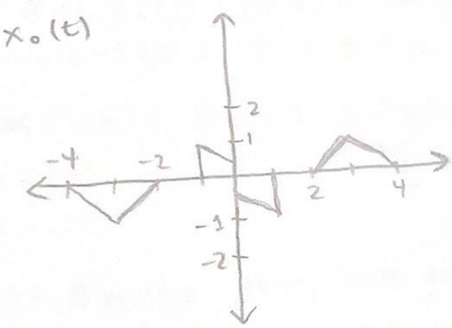
(a) (8 pts) Sketch even and odd components of $x(t)$. Label the value of the even and odd components at $t = 0$. Assume $x(0) = 1$.

(b) (7 pts) Sketch $-2x(-\frac{1}{2}t + 1)$.

a) $x_e(t)$



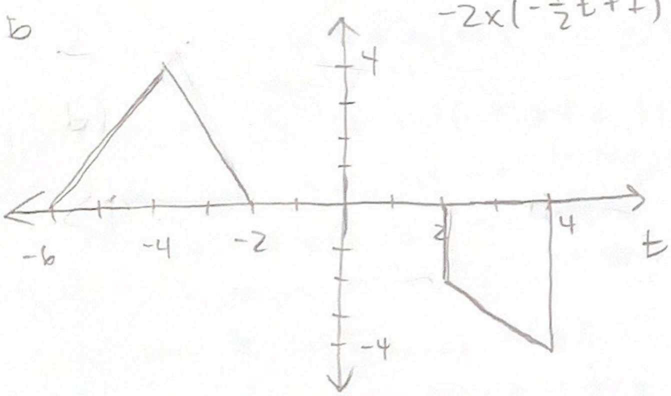
$x_o(t)$



$-2x(-\frac{1}{2}t+1)$

b

$-2x(-\frac{1}{2}t+1)$



Problem 2 (13 pts)

The system S is given by the following relation

$$y(t) = x(t) + e^{-t}x(t)u(t+1), \quad -\infty < t < \infty$$

Using the system relation, answer the following questions about the system. You need to justify your answer and show your work.

(a) (3 pts) Is the system linear? Prove it.

(b) (3 pts) Is the system time-varying or time-invariant? Justify your answer.

(c) (2 pts) Is the system causal or not causal? Justify your answer.

(d) (4 pts) Find the impulse response function of the system.

a) $S\{\alpha x_1(t) + \beta x_2(t)\} = \alpha x_1(t) + \beta x_2(t) + e^{-t}[\alpha x_1(t) + \beta x_2(t)]u(t+1)$

$= \alpha x_1(t) + \beta x_2(t) + e^{-t}u(t+1)\alpha x_1(t) + e^{-t}u(t+1)\beta x_2(t)$

$= \alpha x_1(t) + e^{-t}u(t+1)\alpha x_1(t) + \beta x_2(t) + e^{-t}u(t+1)\beta x_2(t)$

$\alpha y_1(t) = \alpha x_1(t) + e^{-t}u(t+1)\alpha x_1(t)$

$\beta y_2(t) = \beta x_2(t) + e^{-t}u(t+1)\beta x_2(t)$

$S\{\alpha x_1(t) + \beta x_2(t)\} = \alpha y_1(t) + \beta y_2(t)$ therefore the system is $\boxed{\text{linear}}$ ✓

b) $z(t) = S\{x(t-t_0)\} = x(t-t_0) + e^{-t}x(t-t_0)u(t+1)$

$y(t-t_0) = x(t-t_0) + e^{-(t-t_0)}x(t-t_0)u(t-t_0+1)$

$z(t) \neq y(t-t_0)$ therefore the system is $\boxed{\text{TV}}$

c) the system is $\boxed{\text{causal}}$ It only relies on the input $x(t)$ up to time t.

d) $\delta(t-\tau) \rightarrow \boxed{S} \longrightarrow h(t,\tau)$

$h(t,\tau) = \delta(t-\tau) + e^{-t}\delta(t-\tau)u(t+1)$

$\boxed{h(t,\tau) = \delta(t-\tau)(1 + e^{-t}u(t+1))}$

Problem 3 (15 pts)

The impulse response function for a linear system is given below:

$$h(t,\tau) = \sqrt{\frac{1}{\pi}}e^{-t^2+2t\tau-\tau^2}[u(t-\tau+2) - u(t-\tau-2)]$$

(a) (2 pts) Is the system time varying or time invariant? Justify your answer.

(b) (2 pts) Is the system causal or not causal? Justify your answer.

(c) (4 pts) Is this system BIBO stable? Justify your answer.

(d) (3 pts) Write the system input-output relation with $x(t)$ as the input and $y(t)$ as the output. Simplify your answer so it does not include unit step functions.

(e) (4 pts) Compute the step response of this system.

Hint:

$$\int_a^b e^{-x^2}dx = \int_{t+a}^{t+b} e^{-(t-x)^2}dx = \frac{\sqrt{\pi}}{2}(f(b) - f(a))$$

where $f(x)$ is the "error function", defined for all x with output range $[-1,1]$. Thus, this integral will evaluate to a value in the range $[-\sqrt{\pi}, \sqrt{\pi}]$ for any a and b. You do not need to simplify error functions.

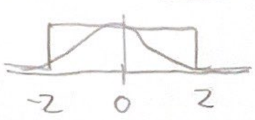
a) $h(t,\tau) = \sqrt{\frac{1}{\pi}}\, e^{-(t^2-2t\tau+\tau^2)}[u(t-\tau+2) - u(t-\tau-2)]$

$h(t,\tau) = \sqrt{\frac{1}{\pi}}\, e^{-(t-\tau)^2}[u(t-\tau+2) - u(t-\tau-2)]$

$h(t-\tau,0) = h(t,\tau)$ therefore $h(t,\tau)$ only depends on $t-\tau$

so it is $\boxed{TI}$

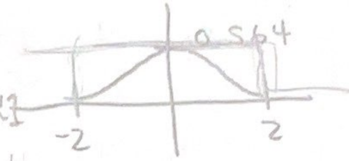
b)



When $\tau=0$, $h(t,\tau) = h(t)$ is nonzero at times that $t<0$, therefore the system is $\boxed{\text{non-causal.}}$

c) $\int_{-\infty}^{\infty} \left| \sqrt{\frac{1}{\pi}}\, e^{-(t)^2} \left[u(t+2) - u(t-2) \right] \right| dt$

$= \int_{-2}^{2} \sqrt{\frac{1}{\pi}}\, e^{-(t)^2}\, dt \approx 1$



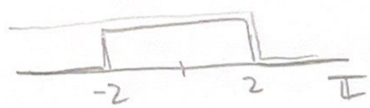
$= \frac{1}{2} f(+2) - \frac{1}{2}(-2) \approx 1$

$\frac{1}{2}\left(f(+2) - f(2) \right) < \infty$

$= u(-t+2)\, u(t+2)$

because $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$, the system is BIBO stable

d) $\int_{-\infty}^{\infty} u(t-\tau) \sqrt{\frac{1}{\pi}}\, e^{-\tau^2} \left[u(-\tau+2)\, u(\tau+2) \right]$



if $t < -2$, $-2 \le t < 2$, $t > 2$

if $t < -2$, 0

if $-2 \le t < 2$ $\quad \int_{-2}^{t} \sqrt{\frac{1}{\pi}}\, e^{-\tau^2}\, d\tau$

if $t \geq 2$ $\quad \int_{-2}^{2} \sqrt{\frac{1}{\pi}}\, e^{-\tau^2}\, d\tau$

$\int \sqrt{\frac{1}{\pi}}\, e^{-\tau^2}\, d\tau \quad \sqrt{\frac{1}{\pi}} \int_{-2}^{t} e^{-\tau^2}\, d\tau$

$= \sqrt{\frac{1}{\pi}} \left(f(t) - f(-2) \right)$

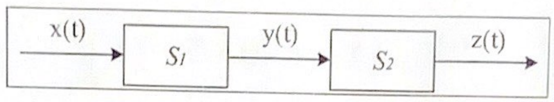
$\cancel{\sqrt{\frac{1}{\pi}}}\, \cancel{\frac{\sqrt{\pi}}{2}} = \frac{1}{2}$ $\quad = \sqrt{\frac{1}{\pi}} \left(f(2) - f(-2) \right)$

$f(x)$ = error function

$$\boxed{S(t) = \begin{cases} 0, & t < -2 \\ \frac{1}{2}\left(f(t) - f(-2) \right), & -2 \le t < 2 \\ \sqrt{\frac{1}{2}} \left(f(2) - f(-2) \right), & t \geq 2 \end{cases}}$$

Problem 4 (20 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.



The first system S_1 is described by:

$$y(t) = \int_{-\infty}^{\infty} \delta(t-\sigma-1)x(\sigma) - \delta(t-\sigma-3)x(\sigma)d\sigma$$

where $x(t)$ and $y(t)$ are the input and the output, respectively. The second system is described by:

$$z(t) = \int_{t-1}^{t} y(\sigma)e^{t-\sigma}d\sigma$$

where $y(t)$ and $z(t)$ and the input and the output, respectively.

(a) (6 pts) Find the impulse response functions $h_1(t,\tau)$ and $h_2(t,\tau)$ of S_1 and S_2 respectively.

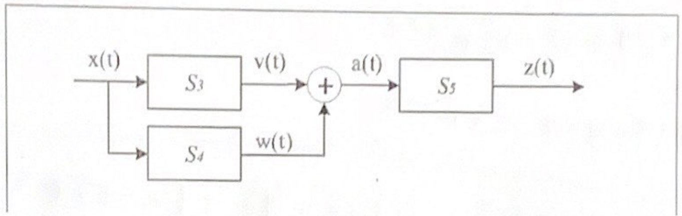
(b) (6 pts) Find the impulse response function $h_{12}(t,\tau)$ of the cascaded system S_{12}. Show your work.

(c) (4 pts) We can express this same cascaded system as a parallel cascaded system, as shown in the figure below. Determine the input-output relationships for each of the systems in this topology, S_3, S_4, and S_5. Do not leave any unit impulse or unit step functions in your equations. Describe how you arrived at this solution.

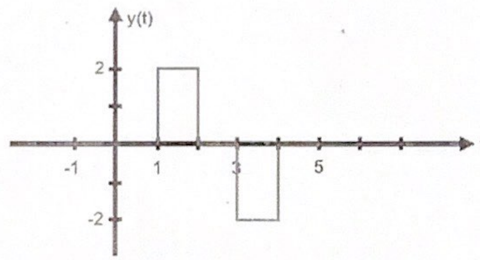
c) $S_3: v(t) = \int_{-\infty}^{\infty} \delta(-t+\sigma+1)x(\sigma)d\sigma = x(t-1)$

$S_4: w(t) = -\int_{-\infty}^{\infty} \delta(-t+\sigma+3)x(\sigma)d\sigma = x(t-3)$

$S_5: \int_{t-1}^{t} y(\sigma)e^{t-\sigma}d\sigma$



(d) (2 pts) While using this system, we probed $y(t)$ and found the following signal. What was the input $x(t)$?

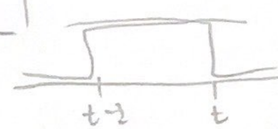


a) $h_1(t,\tau) = \int_{-\infty}^{\infty} \delta(t-\sigma-1)\,\delta(\sigma-\tau) - \delta(t-\sigma-3)\,\delta(\sigma-\tau)\,d\sigma$

$= \int_{-\infty}^{\infty} \delta(t-\sigma-1)\,\delta(\sigma-\tau)\,d\sigma - \int_{-\infty}^{\infty} \delta(t-\sigma-3)\,\delta(\sigma-\tau)\,d\sigma$

$\boxed{h_1(t,\tau) = \delta(t-\tau-1) - \delta(t-\tau-3)}$

$h_2(t,\tau) = \int_{t-1}^{t} \delta(\sigma-\tau)\,e^{t-\sigma}\,d\sigma$



$u(\sigma-t+1)\,u(-\sigma+t)$

$= \int_{-\infty}^{\infty} u(\sigma-t+1)\,u(-\sigma+t)\,e^{t-\sigma}\,\delta(\sigma-\tau)\,d\sigma$

$= u(\tau-t+1)\,u(-\tau+t)\,e^{t-\tau}$

$\boxed{h_2(t,\tau) = u(-(t-\tau)+1)\,u(t-\tau)\,e^{t-\tau}}$

both rely only on $t-\tau$

$$\boxed{\begin{aligned} h_1(t) &= \delta(t-1) - \delta(t-3) \\ h_2(t) &= u(-t+1)\,u(t)\,e^{t} \end{aligned}}$$

b) $\int_{-\infty}^{\infty} [\delta(\tau-1) - \delta(\tau-3)]\, u(-(t-\tau)+1)\, u(t-\tau)\, e^{t-\tau}$

$\int_{-\infty}^{\infty} \delta(\tau-1)\, u(-(t-\tau)+1)\, u(t-\tau)\, e^{t-\tau}\, d\tau$

$= u(-t+1+1)\, u(t-1)\, e^{t-1}$

$u(-t+2)\, u(t-1)\, e^{t-1}$

$\int_{-\infty}^{\infty} - u(-t+3+1)\, u(t-3)\, e^{t-3}$

$\boxed{h_{12}(t) = u(-t+2)\, u(t-1)\, e^{t-1} - u(-t+4)\, u(t-3)\, e^{t-3}}$

Problem 5 (8 pts) Find the Laplace transform and ROC of the following functions. Show all of your work and specify any Laplace identities or properties you use.

(a) (3 pts)

$$f(t) = sin(5t - 15)sin(3t - 9)u(t - 3) + u(t)$$

Hint: $sin(\theta)sin(\phi) = \frac{1}{2}[cos(\theta - \phi) - cos(\theta + \phi)]$.

(b) (5 pts)

$$f(t) = \int_{0^-}^{\infty} e^{-\sigma} sin(2\sigma - 8)u(\sigma - 4)u(t - \sigma)d\sigma$$

a) $f(t) = \frac{1}{2}(\cos(5t - 15 - 3t + 9) - \cos(5t - 15 + 3t - 9)]\, u(t-3) + u(t)$

$= \frac{1}{2}[\cos(2t - 6) - \cos(8t - 24)]\, u(t-3) + u(t)$

$= \underbrace{\frac{1}{2}[\cos(2(t-3)) - \cos(8(t-3))]\, u(t-3)}_{*} + u(t)$

$Ls\{u(t)\} = \frac{1}{s}$

$* = \frac{1}{2}[\cos(2(t-3)) - \cos(8(t-3))]\, u(t-3)$

$\cos(\Omega_0 t)u(t) \xrightarrow{Ls} \frac{s}{s^2 + \Omega_0^2}$

$\xrightarrow{Ls} \frac{1/2\, s}{s^2 + 4} - \frac{1/2\, s}{s^2 + 64}$

time shift: $f(t - \alpha) \to e^{-\alpha s}F(s)$

$\alpha = 3$

$$\boxed{F(s) = \frac{e^{-3s}s}{2(s^2+4)} - \frac{e^{-3s}s}{2(s^2+64)} + \frac{1}{s}, \quad Re\{s\} > 0}$$

b) $f(t) = \int_{0^-}^{\infty} e^{-\sigma} \sin(2\sigma - 8)\, \underbrace{u(\sigma-4)\, u(t-\sigma)}\, d\sigma$

I would use

$\int_{0^-}^{t} f(\tau)\, d\tau \xrightarrow{LS} \frac{F(s)}{s}$ property

Use this to make integral of t w/ window

Problem 6 (12 pts)

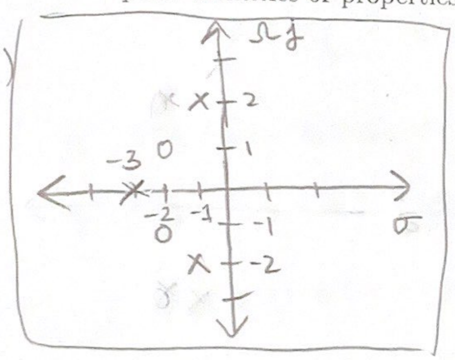
The transfer function for a causal system S is:

$$H(s) = \frac{3s^2 + 12s + 15}{(s^2 + 2s + 5)(s + 3)}$$

(a) (4 pts) Sketch the zero-pole plot of $H(s)$. Describe how you found the poles and zeros.

(b) (8 pts) Find the impulse response function of system S by finding the Inverse Laplace transform of $H(s)$. Show your work and specify any Laplace identities or properties you use.

a)



$3(s^2+4s+5)$

zeros: $\dfrac{-4\pm\sqrt{16-4(5)}}{2}$

$\dfrac{-4\pm\sqrt{-4}}{2}$

$\dfrac{-\cancel{4}\pm\cancel{2}j}{\cancel{2}}$

quad eqn:

$-2\pm j$

s^2+2s+5

poles: $\dfrac{-2\pm\sqrt{4-(4)(5)}}{2}$

quad eqn:

$\dfrac{-2\pm\sqrt{-16}}{2}$

$-1\pm 2j$

$s+3$

$s=-3 \rightarrow -3$

b) $\frac{3s^2+12s+15}{(s^2+2s+5)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+5}$

$$\left.\frac{3s^2+12s+15}{s^2+2s+5}\right|_{s=-3} = \frac{3(9)+12(-3)+15}{9-6+5} = \frac{27-36+15}{9-6+5} = \frac{6}{8} = \frac{3}{4}$$

$$\boxed{A = \frac{3}{4}}$$

$$= \cancel{As^2} + \cancel{A2s} + \cancel{A5} + \cancel{Bs^2} + \cancel{B3s} + \cancel{Cs} + \cancel{C3}$$

$$3s^2+12s+15 = s^2(A+B) + s(A2+B3+C) + C3 + A5$$

$$A + \cancel{8}B = 3$$

$$\frac{3}{4} + B = 3 \qquad 3-\frac{3}{4} \qquad \frac{12}{4} - \frac{3}{4} = \frac{9}{4}$$

$$\boxed{B = \frac{9}{4}}$$

$$5 \times \frac{3}{4} + 3 \times C = 15$$

$$\frac{15}{4} + 3 \times C = 15$$

$$\frac{45}{4}$$

$$\boxed{C = \frac{15}{4}}$$

$$\frac{6}{4} + \frac{27}{4} + \frac{15}{4} =$$

$$= \frac{3/4}{s+3} + \frac{9/4\,s + 15/4}{s^2+2s+5}$$

$$\frac{3/4}{s+3} + \frac{9/4\, s + 15/4}{s^2+2s+5}$$

$s^2+2s+5 =$

$\frac{2}{2}$ $\quad d=1$ $\quad c = 5 - \frac{4}{4} = 4$

$(s+1)^2 + 4$

$$\frac{9/4(s+1-1)}{(s+1)^2+4}$$

$e^{-at}\cos(\Omega_0 t)u(t) = \underbrace{\frac{s+a}{(s+a)^2+\Omega_0^2}}_{\text{identity}}$

$$9/4\frac{(s+1)}{(s+1)^2+4} - \frac{9/4}{(s+1)^2+4} + \frac{15/4}{(s+1)^2+4}$$

$$\frac{9/4(s+1)}{(s+1)^2+4} \quad \frac{+3/4 \cdot \frac{2}{1}\frac{1}{2}}{(s+1)^2+4}$$

$$\boxed{H(s) = \frac{3}{4}e^{-3t}u(t) + \frac{9}{4}e^{-t}\cos(2t)u(t) + \frac{3}{4}e^{-t}\sin(2t)u(t)}$$

$e^{-at}\sin(\Omega_0 t)u(t) = \frac{\Omega}{(s+a)^2+\Omega_0^2}$ } identity