Discussion: 1C

UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm Examination I
February 6, 2018
Duration: 1 hr 50 mins.

INSTRUCTIONS:

- The exam has 6 problems and 14 pages.
- The exam is closed-book.
- One cheat sheet of A4 size is allowed.
- Calculator is NOT allowed.
- Write your discussion session in the top-right corner. ↗↗

Your name:– ________

Student ID:– ________

Table 1: Score Table

Problem	a	b	c	d	Score	
1	4	4	4		12	12
2	2	2	8		12	12
3	6	2	4	6	18	18
4	8	8			16	9
5	10	6			16	15
6	5	5	6		16	9
Total					90	75

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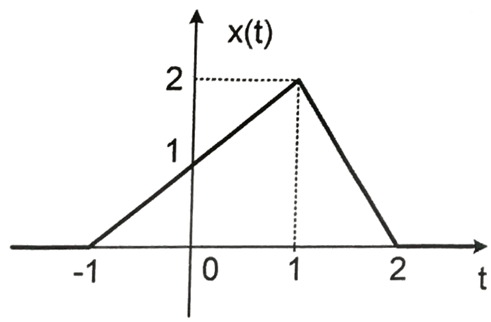
Problem 1 (12 pts) Consider the following signal $x(t)$ for (a), (b)



(a) (4 pts) Sketch even and odd decompositions $x_e(t)$ and $x_o(t)$.

(b) (4 pts) Sketch $x(-2t+3)$.

(c) (4 pts) Sketch $x(t/3+2)$.

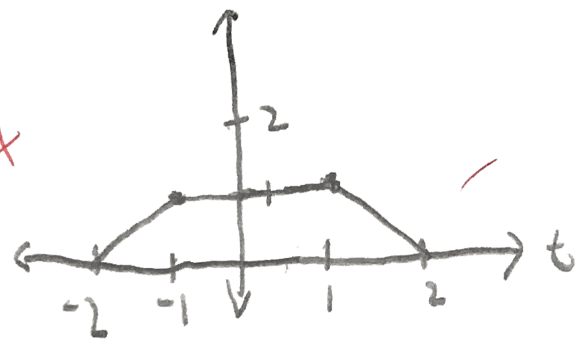
$x(-t)$:

a. $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

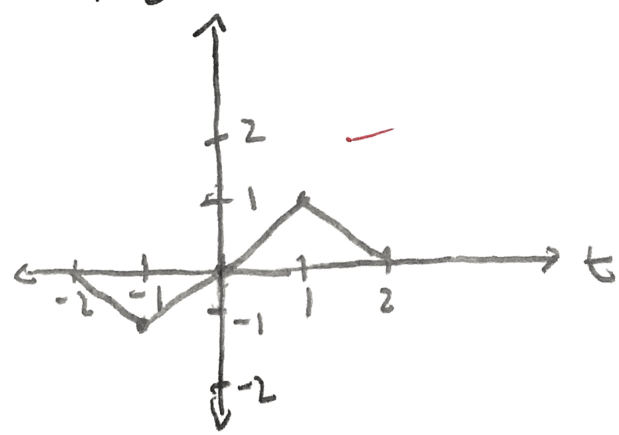
$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$x_e(t)$

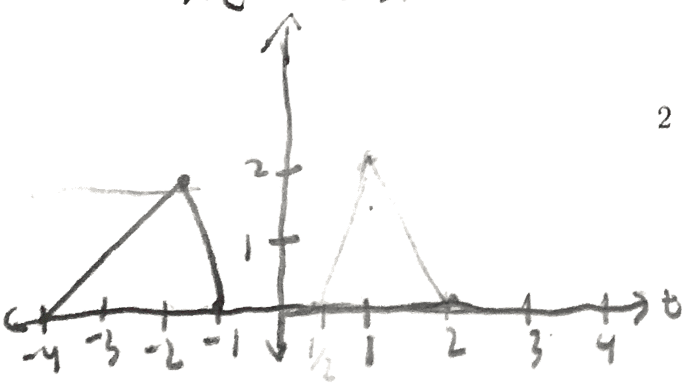
4



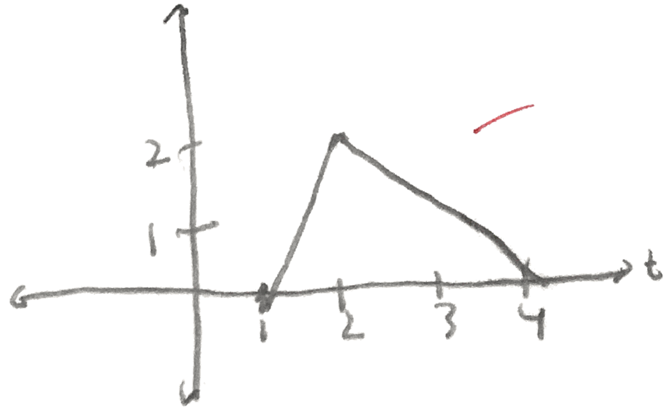
$x_o(t)$



b. $x(-t+3)$



$x(-t+3)$



b. cont.

$x(-2t+3)$

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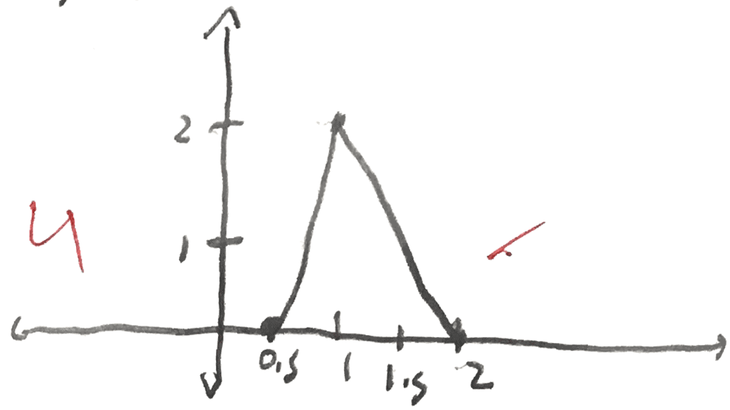
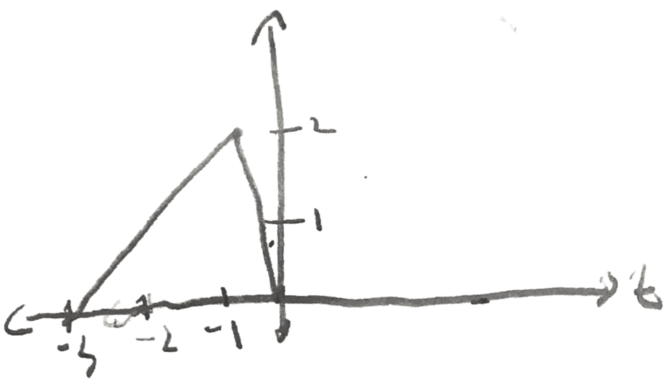


c. $x(t+2)$



$x(\frac{1}{3}t+2)$

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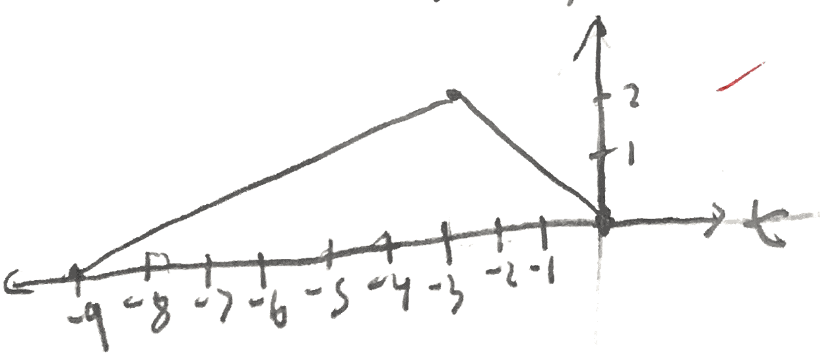

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Problem 2 (12 pts) In this problem, we identify system properties from the impulse response function:

$$h(t,\tau) = e^{-(t-\tau)}u(t-\tau)u(t) \tag{1}$$

(a) (2 pts) Is the system TV or TI? Explain.

(b) (2 pts) Is it C or NC? Explain.

(c) (8 pts) Find the output $y(t)$ if the input is $x(t) = (t-2)u(t-2)$.

a. $h(t,\tau)$ cannot be expressed as a function of $t-\tau$. Therefore, the system is $\boxed{TV.}$

2

b. $h(t,\tau)u(t-\tau) = e^{-(t-\tau)}u(t-\tau)u(t)u(t-\tau)$
$= e^{-(t-\tau)}u(t-\tau)u(t)$ since $u(t-\tau)u(t-\tau) = u(t-\tau)$
$= h(t,\tau)$.

2

This means that $h(t,\tau) = 0$ for $t<\tau$, which means the $\boxed{\text{system is causal (C)}}$

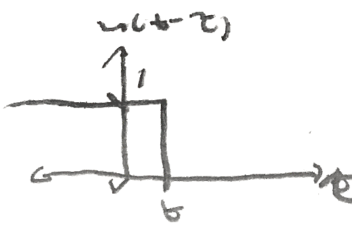
c. $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\,d\tau = \int_{-\infty}^{\infty}(\tau-2)u(\tau-2)e^{-(t-\tau)}u(t-\tau)u(t)\,d\tau$

$u(\tau-2)$ ⇓ 0 if $\tau<2$, 1 o.w.

$= \int_{2}^{\infty}(\tau-2)e^{-(t-\tau)}u(t-\tau)u(t)\,d\tau$

if $t<2$:

$\int_{2}^{\infty}(\tau-2)e^{-(t-\tau)}u(t-\tau)u(t) = 0$



If $t \geq 2$:

$$\int_2^{\infty} (\tau-2)e^{-(t-\tau)} u(t-\tau)u(t)\, d\tau$$

$$= \int_2^{t} (\tau-2)e^{-(t-\tau)}\, d\tau$$

$$= \int_2^t \tau e^{\tau-t}\, d\tau - \int_2^t 2e^{-(t-\tau)}\, d\tau$$

$$\int_2^t \underset{u}{\tau}\, \underset{dv}{e^{\tau-t}}\, d\tau = \tau e^{\tau-t}\Big|_2^t - \int_2^t e^{\tau-t}\, d\tau =$$

$$= te^0 - 2e^{-(t-2)} - e^{\tau-t}\Big|_2^t$$

$$= t - \cancel{2}e^{-(t-2)} - 1$$

$$\int_2^t 2e^{-(t-\tau)}\, d\tau = \quad 2e^{-(t-\tau)}\Big|_2^t = 2 - 2e^{-(t-2)}$$

$$\int_2^t (\tau-2)e^{-(t-\tau)}\, d\tau = \quad t + e^{-(t-2)} - 3$$

$$y(t) \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\, d\tau = \begin{cases} t + e^{-(t-2)} - 3 & t \geq 2 \\ 0 & t < 2 \end{cases}$$

8

$$\boxed{y(t) = \left[t + e^{-(t-2)} - 3\right] u(t-2)}$$

18

Problem 3 (18 pts)
Consider IPOP relation for an LTI system S:

$$y(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}\left[\cos(t)\cos(\tau) + \sin(t)\sin(\tau)\right]x(\tau)d\tau$$

where $x(t)$ and $y(t)$ are input and output of the system, respectively.

(a) (6 pts) Find the impulse response function $h(t)$.

(b) (2 pts) Is the system C or NC? Provide justification.

(c) (4 pts) Is this system BIBO stable? Provide justification.

(d) (6 pts) Find Laplace transform $H(s)$ and ROC.

(Hint: Use the identity $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$)

a. $x(t) = \delta(t)$

$h(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos(t)\cos(\tau) + \sin(t)\sin(\tau)]\,\delta(\tau)\,d\tau$

$= e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos(t-\tau)]\,\delta(\tau)\,d\tau$

$= e^{-t}\int_{-\infty}^{t} e^{0}\cos(t-0)\,\delta(\tau)\,d\tau$

6

$= e^{-t}\,\cancel{e^{0}}_{1}\cos(t)\int_{-\infty}^{t}\delta(\tau)\,d\tau \Rightarrow \begin{cases} 1 & \text{if } t>0 \\ 0 & \text{if } t<0 \end{cases}$

$= e^{-t}\cos(t)\,u(t)$ $\boxed{h(t) = e^{-t}\cos(t)u(t)}$

b. $\boxed{\text{The system is C (causal)}}$ This is because

2

$h(t)\cdot u(t) = e^{-t}\cos(t)u(t)u(t)$
$= e^{-t}\cos(t)u(t) = h(t)$ since $u(t)\cdot u(t) = 0$

In other words, when $t<0$, $h(t) = 0$ This means the system is causal.

c. $$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} e^{-t}\cos(t)u(t)\,dt$$

$$= \int_0^{\infty} e^{-t}\cos(t)\,dt = e^{-t}\sin(t)\Big|_0^{\infty}$$

$$= e^{-\infty}\sin(\infty) - e^{-0}\sin(0)$$

$$= 0 - 0 = 0 < \infty$$

4

since $\int_{-\infty}^{\infty} h(t)\,dt < \infty$, the $\boxed{\text{system is BIBO stable}}$

d. $h(t) = e^{-t}\cos(t)u(t)$

Note: $\mathcal{L}_s\{\cos(t)u(t)\} = \dfrac{s}{s^2+1}$

from the frequency shifting property:

if $f(t) \xrightarrow{\mathcal{L}_s} F(s)$

then $e^{-at} f(t) \xrightarrow{\mathcal{L}_s} F(s+a)$

therefore

$$H(s) = \mathcal{L}_s\{e^{-t}\cos(t)u(t)\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

$$\boxed{H(s) = \frac{s+1}{s^2+2s+2}}$$

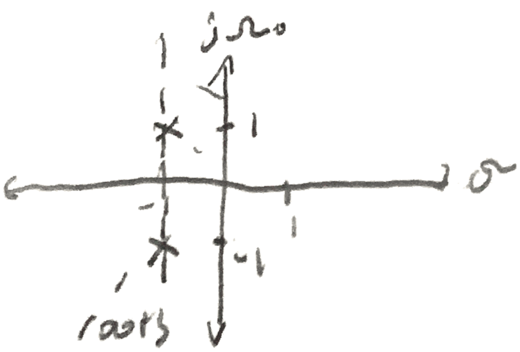
consider denominator polynomial:

s^2+2s+2 : find roots: $s = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm \sqrt{-4}}{2}$

6

$= \dfrac{-2 \pm 2j}{2} = -1 \pm j$

$s = \sigma + j\Omega_0$



rightmost root is at

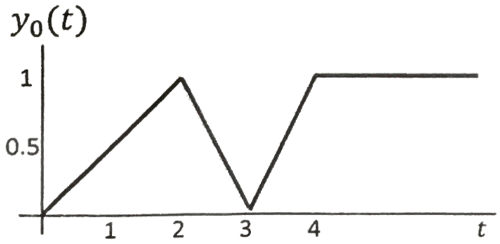
$\sigma = -1$

therefore the ROC is:

$$\boxed{\text{ROC: } \mathrm{Re}\{s\} > -1}$$

9

Problem 4 (16 pts) Consider an LTI system S_0 with input $x_0(t)$ and the impulse response function $h_0(t)$. The corresponding output $y_0(t)$ is shown below:



8 (a) (8 pts) Consider an LTI system S_1 with input $x_1(t) = x_0(t+2)$ and IRF $h_1(t) = h_0(t-1)$. Express the output $y_1(t)$ as a function of $y_0(t)$ and then plot it.

1 (b) (8 pts) Consider an LTI system S_2 with input $x_2(t) = x_0(-t)$ and IRF $h_2(t) = h_0(-t)$. Express the output $y_2(t)$ as a function of $y_0(t)$ and then plot it.

a. $Y_1(t) = \int_{-\infty}^{\infty} x_1(\tau) h_1(t-\tau)\, d\tau$

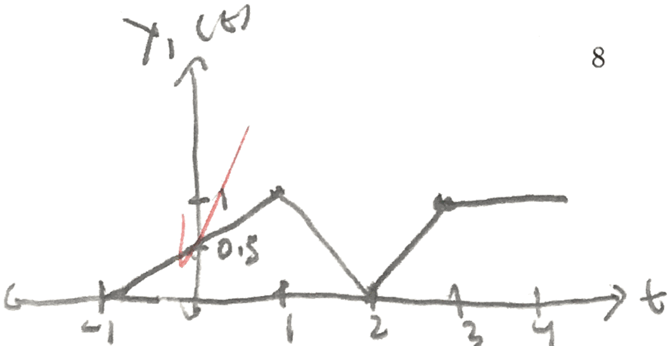
$= \int_{-\infty}^{\infty} x_0(\tau+2)\, h_0((t-1)-\tau)\, d\tau$ set $T = t-1$

$= \int_{-\infty}^{\infty} x_0(\tau+2)\, h_0(T-\tau)\, d\tau$

$= y_0(T+2)$ since system is LTI

$= y_0((t-1)+2) = y_0(t+1)$

$\boxed{y_1(t) = y_0(t+1)}$



b. $y_2(t) = \int_{-\infty}^{\infty} x_2(\tau)\, h_2(t-\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} x_0(-\tau)\, h_0(-(t-\tau))\, d\tau$$

1

$$= \int_{-\infty}^{\infty} x_0(-\tau)\, h_0(-t+\tau)\, d\tau$$

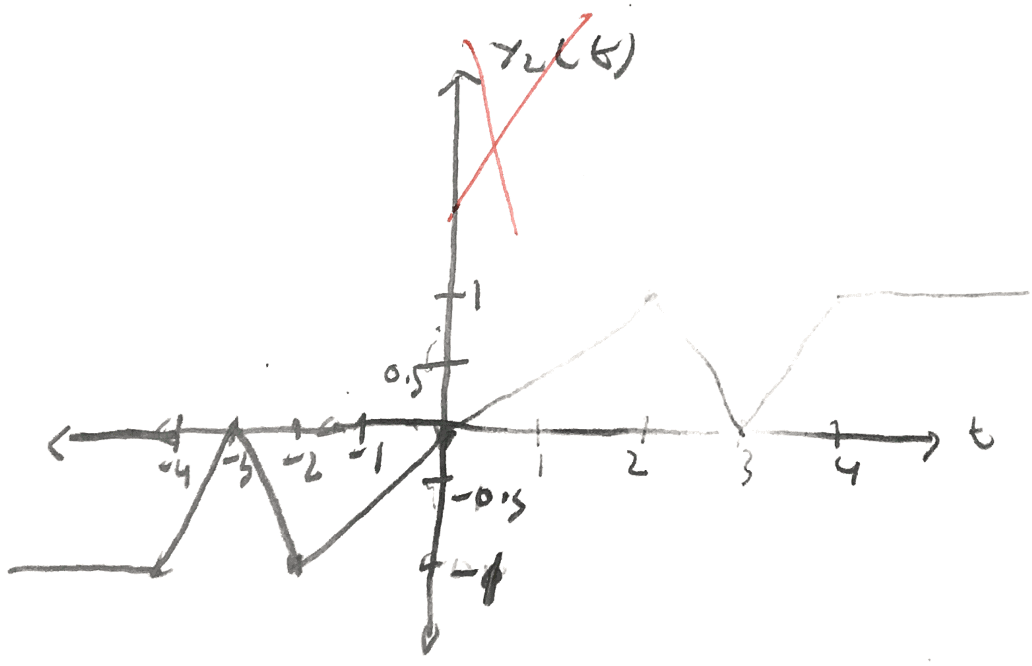
let $T = -t$

if $p = -\tau$

$dp = -d\tau$

$$= -\int_{+\infty}^{-\infty} x_0(p)\, h_0(T-p)\, dp = -y_0(T) = -y_0(-t)$$

$$\boxed{y_2(t) = -y_0(-t)}$$



15 **Problem 5** (16 pts) Consider a cascade combination of two systems S_1 and S_2: $x(t)$ is input to S_1 and $y(t)$ is the output, while the output of S_2 is $z(t)$.

$$x(t) \rightarrow [S_1] \rightarrow y(t) \rightarrow [S_1] \rightarrow z(t)$$

The IPOP relation for S_1 and S_2 is:

$$S_1 : y(t) = e^{-t}x(t)u(t),$$

$$S_2 : z(t) = \int_0^t e^{-(t-\sigma)}y(\sigma)u(\sigma)d\sigma.$$

10 (a) (10 pts) Compute impulse response function $h_{12}(t,\tau)$ of the cascaded system S_1S_2.

5 (b) (6 pts) Compute the output $z(t)$ if the input is $x(t) = e^{-3t}[u(t) - u(t-3)]$.

S_1: $x(t) = \delta(t-\tau)$

a. $h_1(t,\tau) = e^{-t}\delta(t-\tau)u(t)$ signal only matters for $t \geq 0$, 0 o.w. and $\tau \geq 0$.

$h_{12}(t,\tau) = S_2[h_1(t,\tau)]$

$$h_{12}(t,\tau) = \int_0^t e^{-(t-\sigma)} h_1(\sigma,\tau)u(\sigma)\, d\sigma$$

$$= \int_0^t e^{-(t-\sigma)} e^{-\sigma} \delta(\sigma-\tau)\, u(\sigma)\cancel{u(\sigma)}\, d\sigma$$

$$= \int_0^t e^{-(t-\tau)} e^{-\tau} \delta(\sigma-\tau)\, u(\tau)\, d\sigma$$

- consider $t \geq 0$ and $\tau \geq 0$.

$$= e^{-t}u(\tau)\int_0^t \delta(\sigma-\tau)\, d\sigma = e^{-t}u(\tau)u(t-\tau)$$

$\Downarrow$
0 if $t < \tau$
1 if $t > \tau$

$$\boxed{h_{12}(t,\tau) = e^{-t}u(\tau)u(t-\tau)}$$ for $t \geq 0$, $\tau \geq 0$

b. $z(t) = \int_{-\infty}^{\infty} x(\tau)\, h_1(t,\tau)\, d\tau$

$= \int_{-\infty}^{\infty} e^{-3\tau} \underbrace{[u(\tau) - u(\tau-3)]}_{\substack{1 \\ 1 \text{ if} \\ 0 \le \tau \le 3}} e^{-t} u(\tau) u(t-\tau)\, d\tau$

$= \int_{0}^{3} e^{-3\tau} e^{-t} u(t-\tau)\, d\tau = INT$

if $t < 0$

then $INT = 0$

if $0 < t < 3$

then $INT = \int_0^t e^{-3\tau} e^{-t}\, d\tau = e^{-t}\left[-\frac{e^{-3\tau}}{3}\Big|_0^t\right]$

$= e^{-t}\left[\frac{1}{3} - \frac{e^{-3t}}{3}\right]$

if $t > 3$

then $INT = \int_0^3 e^{-3\tau} e^{-t}\, d\tau = e^{-t}\left[-\frac{e^{-3\tau}}{3}\Big|_0^3\right]$

$= e^{-t}\left[\frac{1}{3} - \frac{e^{-9}}{3}\right]$

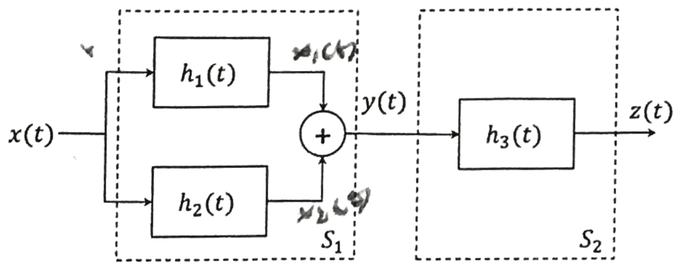
$z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - \frac{e^{-3t}}{3} & \text{if } 0 \le t \le 3 \\ 1 - \frac{e^{-9}}{3} & \text{if } t > 3 \end{cases}$

$\boxed{z(t) = \left[1 - \frac{e^{-3t}}{3}\right]\left[u(t) - u(t-3)\right] + \left[1 - \frac{e^{-9}}{3}\right] u(t-3)}$

9 **Problem 6** (16 pts)
Consider a cascaded LTI system $S_1 S_2$ as follows

$$x(t) \to [S_1] \to y(t) \to [S_2] \to z(t)$$

The cascade system is shown below.



where $h_1(t) = \delta(t-1)$, $h_2(t) = \delta(t-2)$, and $h_3(t) = \delta(t-1) - \delta(t-2) + \delta(t-3)$.
Let $x(t) = 2(u(t) - u(t-2))$, then

5 (a) (5 pts) Find the IPOP between $x(t)$ and $y(t)$. Plot $y(t)$.

3 (b) (5 pts) Write the impulse response of the cascade system $S_1 S_2$.

1 (c) (6 pts) Compute and plot $z(t)$ for the specified input.

a. $x_1(t) = \int_{-\infty}^{\infty} x(\tau)\, h_1(t,\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-1-\tau)\, d\tau$ if $p = t-1$

$= \int_{-\infty}^{\infty} x(\tau)\, \delta(p-\tau)\, d\tau = x(p)$

$= x(t-1)$

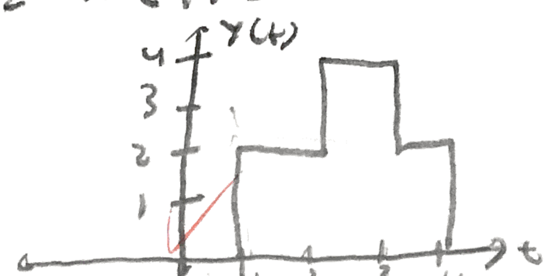
$x_2(t) = \int_{-\infty}^{\infty} x(\tau)\, h_2(t,\tau)\, d\tau$ if $p = t-2$

$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-2-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(\tau)\, \delta(p-\tau)\, d\tau = x(p) = x(t-2)$

$\boxed{y(t) = x(t-1) + x(t-2)}$

since $x(t) = 2[u(t) - u(t-2)]$



b. $h_{1s}(t) = h_1(t) + h_2(t)$

$= \delta(t-1) + \delta(t-2)$

$h_{2s}(t) = \delta(t-1) + \delta(t-2) + \delta(t-3)$

$$h_{12}(t) = \int_{-\infty}^{\infty} h_{1s}(\tau)\, h_{2s}(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} [\delta(\tau-1) + \delta(\tau-2)]\,[\delta(t-\tau-1) + \delta(t-\tau-2) + \delta(t-\tau-3)]\, d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau-1)\delta(t-2) + \delta(\tau-1)\delta(t-3) + \delta(\tau-1)\delta(t-4)\, d\tau$$
$$+ \delta(\tau-2)\delta(t-3) + \delta(\tau-2)\delta(t-4) + \delta(\tau-2)\delta(t-5)\, d\tau$$

$$= \delta(t-2) + 2\delta(t-3) + 2\delta(t-4) + \delta(t-5)$$

$$\boxed{h_{12}(t) = \delta(t-2) + 2\delta(t-3) + 2\delta(t-4) + \delta(t-5)}$$

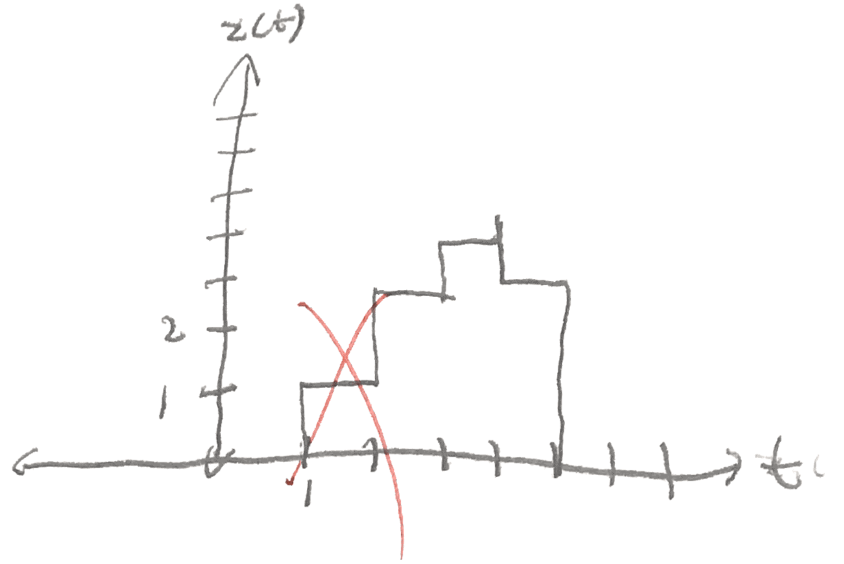
$$z(t) = \int_{-\infty}^{\infty} x(t)\, h_{12}(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} 2[u(\tau) - u(\tau-2)]\, h_{12}(t-\tau)\, d\tau = 2\int_0^2 h_{12}(t-\tau)\, d\tau$$

$$= 2\int_0^2 \delta(t-\tau-2) + 2\delta(t-\tau-3) + 2\delta(t-\tau-4) + \delta(t-\tau-5)\, d\tau$$

$0 \leq t-2 \leq 2$

$$= \boxed{2\big[u(t-2) - u(t-4) + 2(u(t-3) - u(t-5)) + 2(u(t-4) - u(t-6)) + u(t-5) - u(t-7)\big]}$$


z(t)
2
1
1
t

$$+\frac{1}{j2}\cdot I_1 + V_s + I_1 - I_2 + I_1 - I_3 = 0$$

$$\cancel{\left(2 + \frac{1}{2}\right) I_1} = -1$$

$$\left(2 + \frac{1}{2j}\right) I_1 - I_2 - I_3 = -1$$

2. $I_2 \cdot \frac{1}{j8} + (I_2 - I_3) 1 + (I_2 - I_1) \cdot 1 - 1 = 0$

$$-1\, I_1 + \left(2 + \frac{1}{j8}\right) I_2 - I_3 = 1$$

$$I_3 - I_1 + I_3 - I_2 + 2j I_3 = 0$$

$$-I_1 - I_2 + (2 + 2j) I_3 = 0$$