UCLA DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ECE 102: SYSTEMS & SIGNALS

Final Examination
March 19, 2021
Duration: 3 hr 0 min. (+15 min. for Gradescope submission)

INSTRUCTIONS:

- The exam has 6 problems and 23 pages.
- The exam is open-book and open-notes.
- Calculator/MATLAB allowed.
- Show all of your work! No credit given for answers without math steps shown and/or an explanation.
- NO LATE SUBMISSIONS ALLOWED ON GRADESCOPE.

Your name:______________________________

Student ID:______________________________

Table 1: Score Table

Problem	a	b	c	d	e	Score
1	4	1	2	8	4	19
2	2	3	8			13
3	4	1	2	3		10
4	5	6	5	5		21
5	9	5	3	3		20
6	6	6	5	(+2)		17
Total						100

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\vert\alpha\vert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Table 5.1 Basic Properties of the Fourier Transform

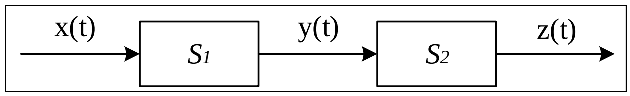
	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}$, $n \geq 1$, integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$
		$\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt$, imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
1	$\delta(t)$	1
2	$\delta(t - \tau)$	$e^{-j\Omega\tau}$
3	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
4	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
5	$\text{sgn}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
6	$A, \quad -\infty < t < \infty$	$2\pi A\delta(\Omega)$
7	$Ae^{-at}u(t), \; a > 0$	$\frac{A}{j\Omega + a}$
8	$Ate^{-at}u(t), \; a > 0$	$\frac{A}{(j\Omega + a)^2}$
9	$e^{-a\lvert t\rvert}, \; a > 0$	$\frac{2a}{a^2 + \Omega^2}$
10	$\cos(\Omega_0 t), \quad -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
11	$\sin(\Omega_0 t), \quad -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
12	$A[u(t + \tau) - u(t - \tau)], \; \tau > 0$	$2A\tau \frac{\sin(\Omega\tau)}{\Omega\tau}$
13	$\frac{\sin(\Omega_0 t)}{\pi t}$	$u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
14	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$

Problem 1 (19 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.



The first system S_1 is causal and described by:

$$y(t) = 7x(t) + x(t-2)$$

where $x(t)$ and $y(t)$ are the input and the output, respectively.

The second system is LTI and described by the following impulse response:

$$h_2(t) = 2(t-1)^3 e^{-5t} u(t-1)$$

(a) (4 pts) Prove S_1 is LTI (i.e. both linear and time-invariant).

(b) (1 pts) Is S_2 causal? Justify your answer.

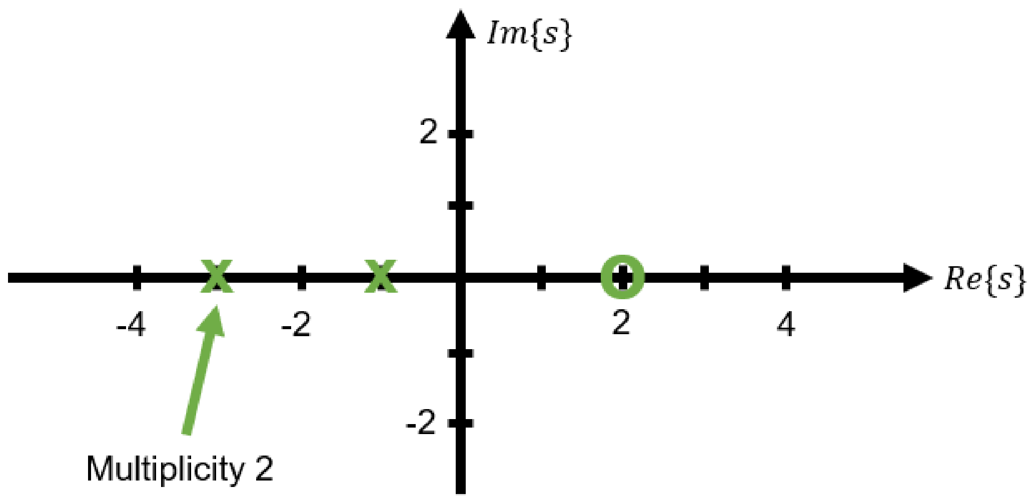
(c) (2 pts) From the IPOP given, find the impulse response functions $h_1(t)$ for S_1. Show your work.

(d) (8 pts) Find the transfer functions $H_1(s)$ and $H_2(s)$ of S_1 and S_2 respectively. Include the ROC of each transfer function. Show your work.

(e) (4 pts) Find the transfer function $H_{12}(s)$ of the cascaded system S_{12}. Include the ROC. Show your work.

Problem 2 (13 pts)

A causal LTI system S has the following pole-zero plot for its transfer function $H(s)$:



Note that a pole with "multiplicity 2" means that there are two instances of that same pole.

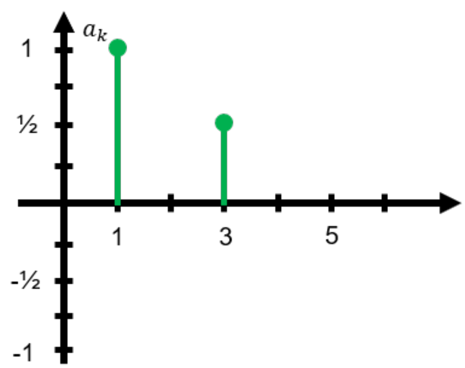
(a) (2 pts) Is the system BIBO stable? Justify your answer.

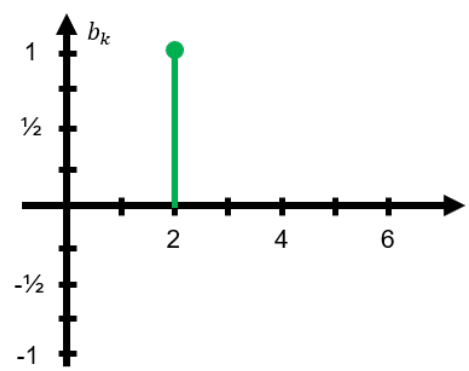
(b) (3 pts) Find the transfer function $H(s)$. Note that $H(0) = 2$. Show your work and label any properties or identities you use.

(c) (8 pts) Find the impulse response function of S, $h(t)$. Show your work and label any properties or identities you use.

Problem 3 (10 pts)

The trigonometric Fourier series coefficients for a periodic signal $x(t)$ are given on plots below. Note that $x(t)$ has a period $T_0 = 1$ and a DC component $X_0 = 2$.





Note: Remember that the trigonometric Fourier series coefficients a_k and b_k represent the signal using the following equation:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) - 2\sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t).$$

(a) (4 pts) What are the complex Fourier series coefficients for $x(t)$? Show your work/justify your answers.

(b) (1 pt) Is $x(t)$ even, odd, or neither? Justify your answer.

(c) (2 pts) Plot the magnitude and phase spectra of $x(t)$. Clearly label your plot.

(d) (3 pts) What was the **power and energy** of the signal $x(t)$? Show your work/justify your answer.

Problem 4 (21 pts)

Joe is stuck on an island and has an emergency beacon that transmits a high frequency sine wave pulse $x(t)$.

$$x(t) = \begin{cases} sin(2\pi \cdot 1000t), & 0 \leq t \leq 1\pi \\ 0 & otherwise \end{cases}$$

(a) (5 pt) First assume that the pulse is only transmitted once. Find the Laplace transform of $x(t)$ and specify its ROC. Show your work and label any properties or identities you use.

You can use $\omega_0 = 2\pi \cdot 1000$ to reduce your writing if you wish.

(b) (6 pts) Find the Fourier transform of $x(t)$. Show your work and label any properties or identities you use.

(c) (5 pts) Now assume that the beacon repeatedly transmits once every 5 seconds. Joe has been stuck on the island literally forever, so this new transmission $x_p(t)$ is truly periodic. Find the Fourier series of $x_p(t)$. Show your work and label any properties or identities you use.

Do not simplify complex exponentials, but do not leave any integrals in your solution.

(d) (5 pts) Can we find the Fourier transform of $x_p(t)$? If we can, find the Fourier transform. Show your work and label any properties or identities you use. If we cannot, explain why.

Problem 5 (20 pts)

The IPOP for a system S with input $x(t)$ and output $y(t)$ is written below:

$$y(t) = \int_{-\infty}^{t} x(\tau) \left(\frac{1}{2} \left[u(t - \tau + 2) - u(t - \tau - 2) \right] \right) d\tau$$

The following signal is applied as input $x(t)$ to system S:

$$x(t) = \frac{sin(2\pi t)}{\pi t}$$

(a) (9 pts) Find the Fourier transform $Y(\omega)$ of the **output** $y(t)$. Show your work and label any properties or identities you use.

(b) (5 pts) Sketch the magnitude and phase spectrum of $Y(\omega)$.

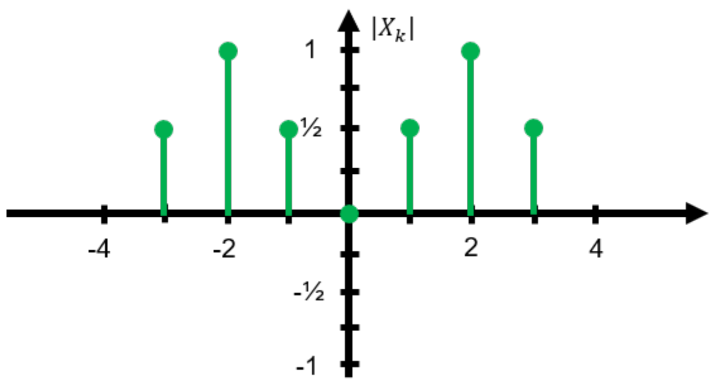
(c) (3 pts) We now sample the output $y(t)$ at $\omega_s = 6\pi$. Sketch the magnitude spectrum of the sampled signal for $10\pi \leq \omega \leq 10\pi$. Label your axes.

(d) (3 pts) What is the minimum sampling rate ω_s for $y(t)$ to avoid aliasing?

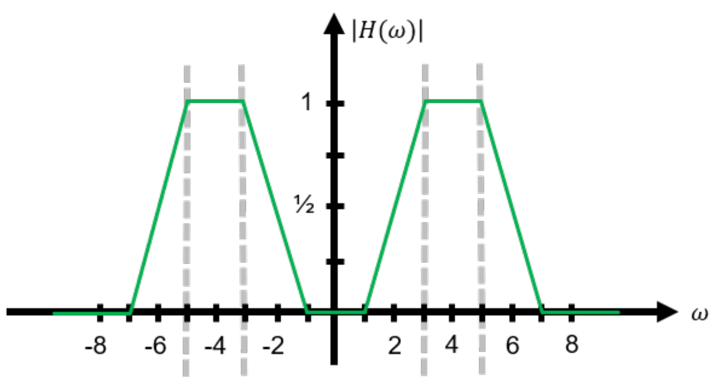
Problem 6 (17 pts)

Gene is back! This time he wants to filter the signal he is measuring to only have frequency components at $\omega = 4$ Hz.

The magnitude spectra of his input periodic signal (with fundamental frequency $\Omega_0 = 2$ Hz) is shown below. Remember that the horizontal axis with respect to integer values k, not ω in Hz.



Gene built a filter with the following frequency response (ω axis in units of Hz):



(a) (6 pts) If Gene uses the filter he built, plot the magnitude spectrum of the output. Label your plot clearly. Does this filter accomplish his goal of only having frequency components at 4 Hz? Explain your answer.

(b) (6 pts) Gene also found 4 more filters in his lab. They are:

1) Ideal Low-Pass filter, cutoff at 1.5 Hz.
2) Ideal Low-Pass filter, cutoff at 2.5 Hz.
3) Ideal High-Pass filter, cutoff at 1.5 Hz.
4) Ideal High-Pass filter, cutoff at 2.5 Hz.

Which two filters can he use to make the filter he needs? Should they be connected in cascade or in parallel? Explain your reasoning and plot the frequency response of your combined filter.

(c) (5 pts) Find the energy of the impulse response for the original filter Gene built, i.e. find the energy of $h(t) = \mathcal{F}^{-1}\{H(\omega)\}$.

(d) (2 EC pts) Challenge question: Find the impulse response of the original filter Gene built, $h(t)$. Show your work and specify any properties or identities you use. Assume the phase of $H(\omega)$ is 0 for all ω.