EE102 Fall 2009
Midterm Exam solution

1 Given the IPOP :

$$\frac{dv(t)}{dt} + 4v(t) = u(t) \qquad t \geq 0$$

a) Find the Initial condition response for zero input

b) Find the response for a unit step input

c) Find the zero state response for

$$u(t) = \operatorname{Sin} 4\pi t, \quad t > 0$$

d) Find the steady state response in case (b)

a) $V(s) = \dfrac{v(0)}{s+4} \to v(t) = v(0)e^{-4t}u(t)$

b) $V(s) = \dfrac{v(0)}{s+4} + \dfrac{1}{s(s+4)} = \dfrac{v(0)}{s+4} + \dfrac{1/4}{s} - \dfrac{1/4}{s+4} \to v(t) = (e^{-4t}(v(0) - 1/4) + 1/4)u(t)$

c)

$$V(s) = \frac{4\pi}{(s^2 + (4\pi)^2)(s+4)} = \frac{\frac{4\pi}{16+16\pi^2}}{s+4} + \frac{-\frac{4\pi}{16+16\pi^2}s + \frac{16\pi}{16+16\pi^2}}{s^2 + (4\pi)^2}$$

$$v(t) = \frac{\pi}{4+4\pi^2}(e^{-4t} - \cos(4\pi t)) + \frac{1}{4+4\pi^2}\sin(4\pi t)$$

d)

$$v(t) = \frac{1}{4}u(t)$$
$$t \to \infty$$

2 Given the IPOP relation :

$$\frac{d^2 v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 17v(t) = u(t)$$

answer questions

a, b, c, and d as in problem 1

(2)
$$\frac{d^2 v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 17\, v(t) = u(t)$$

Taking the Laplace Transform we get,

$$s^2 V(s) - s\,v(0) - v'(0) + 4\left(s\,V(s) - v(0)\right) + 17\, V(s) = U(s)$$

$$V(s)\left(s^2 + 4s + 17\right) = s\,v(0) + v'(0) + 4v(0) + U(s)$$

$$V(s) = \frac{s\,v(0) + v'(0) + 4v(0)}{s^2 + 4s + 17} + \frac{U(s)}{s^2 + 4s + 17}$$

a) $u(t) = 0 \Rightarrow U(s) = 0$

$$V(s) = \frac{s\,v(0) + v'(0) + 4v(0)}{s^2 + 4s + 17} = \frac{s\,v(0)}{(s+2)^2 + (\sqrt{13})^2} + \frac{v'(0) + 4v(0)}{(s+2)^2 + (\sqrt{13})^2}$$

$$= \frac{(s+2)\,v(0)}{(s+2)^2 + (\sqrt{13})^2} + \frac{v'(0) + 4v(0) - 2v(0)}{(s+2)^2 + (\sqrt{13})^2}$$

$$V(s) = \frac{(s+2)\,v(0)}{(s+2)^2 + (\sqrt{13})^2} + \frac{v'(0) + 2v(0)}{\sqrt{13}} \cdot \frac{\sqrt{13}}{(s+2)^2 + (\sqrt{13})^2}$$

Taking the Inverse Laplace Transform, we get

$$\boxed{v(t) = v(0)\, e^{-2t} \cos(\sqrt{13}\, t) + \left(\frac{v'(0) + 2v(0)}{\sqrt{13}}\right) e^{-2t} \sin(\sqrt{13}\, t)}\,, \quad t \geq 0$$

which can also be written in form: $v(t) = (\cdots\cdots)\, e^{-2t} \sin(\sqrt{13}\, t + \psi)$

$\boxed{\text{5 points}}$

b) $u(t)$: Unit Step $\Rightarrow U(s) = \frac{1}{s}$

$$V(s) = \underbrace{\frac{s v(0) + v'(0) + 4v(0)}{s^2+4s+17}}_{\text{Inverse Laplace Transform of this term was found in part (a)}} + \frac{1}{s(s^2+4s+17)}$$

We only need to find the Inverse Laplace Transform of the 2nd term:

$$\frac{1}{s(s^2+4s+17)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+17} \quad , \quad A(s^2+4s+17) + Bs^2 + Cs = 1$$

$$A = \frac{1}{17} \qquad B = \frac{-1}{17} \qquad C = \frac{-4}{17}$$

$$= \frac{A}{s} + \frac{B(s+2)}{(s+2)^2 + (\sqrt{13})^2} + \frac{C-2B}{(s+2)^2+(\sqrt{13})^2}$$

$$= \frac{1}{17}\frac{1}{s} + \left(\frac{-1}{17}\right)\frac{s+2}{(s+2)^2+(\sqrt{13})^2} + \left(\frac{-2}{17}\right)\frac{1}{\sqrt{13}}\frac{\sqrt{13}}{(s+2)^2+(\sqrt{13})^2}$$

The Inverse Laplace Transform of 2nd term is:

$$\left(\frac{1}{17} - \frac{1}{17}e^{-2t}\cos(\sqrt{13}t) - \frac{2}{17\sqrt{13}}e^{-2t}\sin(\sqrt{13}t)\right)$$

Using the above result and the answer found in part (a), we get

$$v(t) = v(0)e^{-2t}\cos(\sqrt{13}t) + \left(\frac{v'(0)+2v(0)}{\sqrt{13}}\right)e^{-2t}\sin(\sqrt{13}t) + \frac{1}{17} - \frac{1}{17}e^{-2t}\cos(\sqrt{13}t) - \frac{2}{17\sqrt{13}}e^{-2t}\sin(\sqrt{13}t) \quad , \quad t \geq 0$$

$$\boxed{v(t) = \frac{1}{17} + \left[v(0) - \frac{1}{17}\right]e^{-2t}\cos(\sqrt{13}t) + \left[\frac{v'(0)+2v(0)}{\sqrt{13}} - \frac{2}{17\sqrt{13}}\right]e^{-2t}\sin(\sqrt{13}t)} \quad , \quad t \geq 0$$

$\boxed{\text{6 points}}$

d) In Steady State, the terms that contain e^{-2t} will be 0.

$$\boxed{V_{ss}(t) = \frac{1}{17}}$$

Steady State Response in case (b).

$\boxed{\text{3 points}}$

c) $v(0)=0$, $v'(0)=0$, $U(s) = \dfrac{4\pi}{s^2+16\pi^2}$

$$V(s) = \frac{4\pi}{(s^2+16\pi^2)(s^2+4s+17)} = \frac{As+B}{s^2+16\pi^2} + \frac{Cs+D}{s^2+4s+17}$$

$$(As+B)(s^2+4s+17) + (Cs+D)(s^2+16\pi^2) = 4\pi$$

s^3 terms : $A + C = 0$

s^2 terms : $B + 4A + D = 0$

s^1 terms : $17A + 4B + 16\pi^2 C = 0$

s^0 terms : $17B + 16\pi^2 D = 4\pi$

We have 4 unknowns and 4 linear equations This could be written in matrix form and be solved using a calculator or computer.

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ 17 & 4 & 16\pi^2 & 0 \\ 0 & 17 & 0 & 16\pi^2 \end{bmatrix}}_{\text{matrix } X} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\pi \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = X^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\pi \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} A &= -0{,}0022 \\ B &= -0{,}0791 \\ C &= 0{,}0022 \\ D &= 0{,}0881 \end{aligned}}$$

$$V(s) = \frac{As+B}{s^2+16\pi^2} + \frac{C(s+2)}{(s+2)^2+(\sqrt{13})^2} + \frac{D-2C}{(s+2)^2+(\sqrt{13})^2}$$

$$v(t) = A\cos(4\pi t) + \frac{B}{4\pi}\sin(4\pi t) + Ce^{-2t}\cos(\sqrt{13}\,t) + \frac{(D-2C)}{\sqrt{13}} e^{-2t}\sin(\sqrt{13}\,t) \quad , \quad t \geq 0$$

Plugging A, B, C & D into the eqn. we get

$$\boxed{v(t) = -0{,}0022\cos(4\pi t) - 0{,}0063\sin(4\pi t) + 0{,}0022\,e^{-2t}\cos(\sqrt{13}\,t) + 0{,}0232\,e^{-2t}\sin(\sqrt{13}\,t)} \quad , \quad t \geq 0$$

which can also be written in form : $v(t) = (\dots)\sin(4\pi t + \Psi) + (\dots)\sin(\sqrt{13}\,t + \Phi)$

$\boxed{\text{6 points}}$

3 What is the system Transform function in problem 2, denote it H (s) ?
Calculate

Log H ($i\omega$), $-\infty < \omega < \infty$

and the gain and phase.

Show that that the gain - or equivalently $|H(i\omega)|$ -
is a maximum at the natural frquency of the system.
What is the value of the maximum ?

$$H(s) = \frac{1}{s^2 + 4s + 17} \to |H(jw)| = \frac{1}{\sqrt{(17 - w^2)^2 + 16w^2}}$$

$$\log(|H(jw)|) = -\frac{1}{2}\log((17 - w^2) + 16w^2)$$

$$\measuredangle H(jw) = -\tan^{-1}\left(\frac{4w}{17 - w^2}\right)$$

$for\ \max\ of\ |H(jw)|$:

$$\frac{d}{dw}|H(jw)| = 0 \to 4w^3 - 36w = 0 \Rightarrow w = \pm 3$$

$$\max |H(jw)| = |H(j3)| = \frac{1}{\sqrt{16 \times 13}} = \frac{1}{4\sqrt{13}}$$