ECE102, Spring 2020 **Final Exam**

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UCLA True Bruin academic integrity principles apply.

This exam is open book, open note and open internet. Collaboration is not allowed.

8:00 am Wednesday, 3 June 2020
- 8:00 am Thursday, 4 June 2020.

Instructions for submission
Submit your work with your answers on this question sheet. Please do not work on a separate paper.

State your assumptions and reasoning for all the questions.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

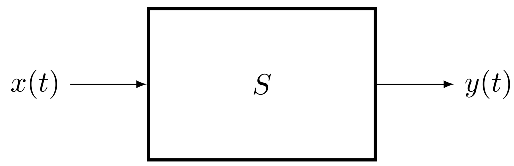
Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 21
Problem 2	_____	/ 24
Problem 3	_____	/ 10
Problem 4	_____	/ 15
Problem 5	_____	/ 15
Problem 6	_____	/ 15
BONUS	_____	/ 6 bonus points
Total	_____	/ 100 points + 6 bonus points

1. **Signal and Systems Basics** (21 points)

(a) (12 points) **System properties.** Consider the following system **S** as shown below where $x(t)$ and $y(t)$ are the input and output pairs of the system, respectively.



For an input $x(t)$, the output $y(t)$ is given by,

$$y(t) = 5x(t+1) + 3\int_{-\infty}^{t-1} cos(t-\tau)x(\tau)d\tau$$

i. Is the system linear? Justify your answer.

Solution:
Linear. Let us assume that inputs $x_1(t)$, $x_2(t)$ to the system produces outputs $y_t(t)$ and $y_2(t)$, respectively. Let us input $ax_1(t) + bx_2(t)$, then

$$\begin{aligned}
y(t) &= 5(ax_1(t+1) + bx_2(t+1)) + 3\int_{-\infty}^{t-1} cos(t-\tau)(ax_1(\tau) + bx_2(\tau))d\tau \\
&= a[5x_1(t+1) + 3\int_{-\infty}^{t-1} cos(t-\tau)x_1(\tau)d\tau] + b[5_2(t+1) + 3\int_{-\infty}^{t-1} cos(t-\tau)x_2(\tau)d\tau] \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

ii. Is the system time invariant? Justify your answer.

Solution:
Time invariant. Let us assume that input $x(t)$ to the system produces $y(t)$ as output. Let us input $x(t+t_0)$ to the system and see its output $y_a(t)$,

$$\begin{aligned}
y_a(t) &= 5x(t+t_0+1) + 3\int_{-\infty}^{t-1} cos(t-\tau)x(\tau+t_0)d\tau \\
&= 5x(t+t_0+1) + 3\int_{-\infty}^{t+t_0-1} cos(t+t_0-\tau)x(\tau)d\tau \\
&= y(t+t_0)
\end{aligned}$$

iii. Is the system casual? Justify your answer.

Solution:
Not causal. The output depends on the value of the input in the future in the term $x(t+1)$.

iv. Find the unit step response, $s(t)$ of the system.

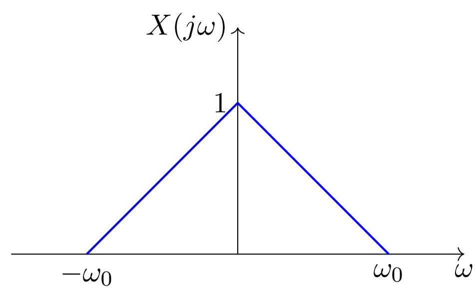
Solution:
Note that the impulse response $h(t)$ is

$$\begin{aligned} h(t) &= 5\delta(t+1) + 3\int_{-\infty}^{t-1} cos(t-\tau)\delta(\tau)d\tau \\ &= 5\delta(t+1) + 3cos(t)\int_{-\infty}^{t-1} \delta(\tau)d\tau \\ &= 5\delta(t+1) + 3cos(t)u(t-1) \end{aligned}$$

For the step response we can use an accumulator:

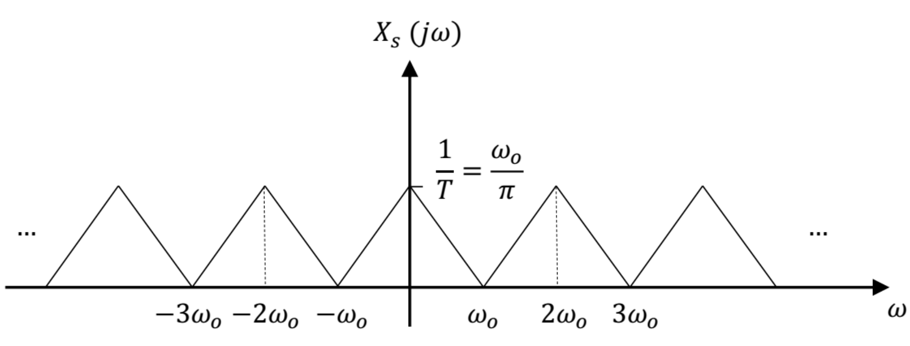
$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \begin{cases} 0 & t < -1 \\ 5 & -1 < t < 1 \\ 5 + 3sin(t) - 3sin(1) & t > 1 \end{cases}$$

(b) (9 points) **Sampling basics.** A signal $x(t)$ has a band-limited spectrum $X(j\omega)$ as shown below, and zero phase at all frequencies. The signal $x(t)$ is sampled by a unit impulse train with a period T to generate a sampled signal $x_s(t)$, with the spectrum $X_s(j\omega)$.



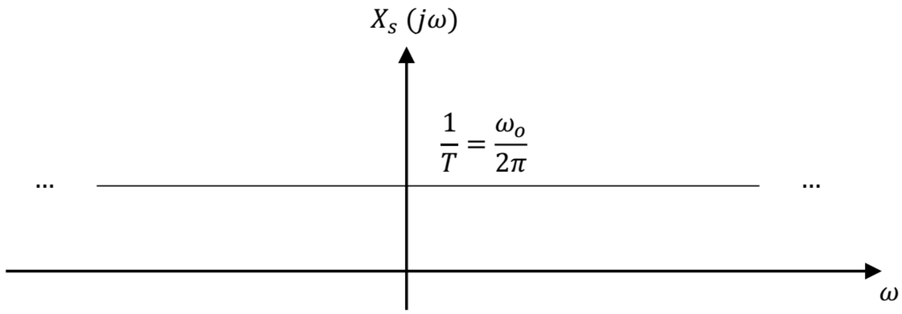
i. Plot the spectrum of $X_s(j\omega)$ when $\frac{2\pi}{T} = 2\omega_0$.

Solution:



ii. Plot the spectrum of $X_s(j\omega)$ when $\frac{2\pi}{T} = \omega_0$.

Solution:

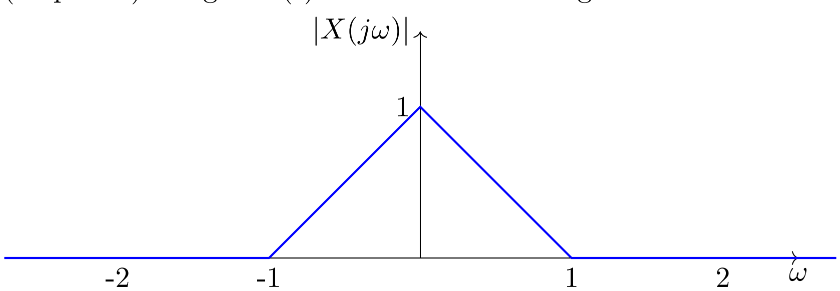


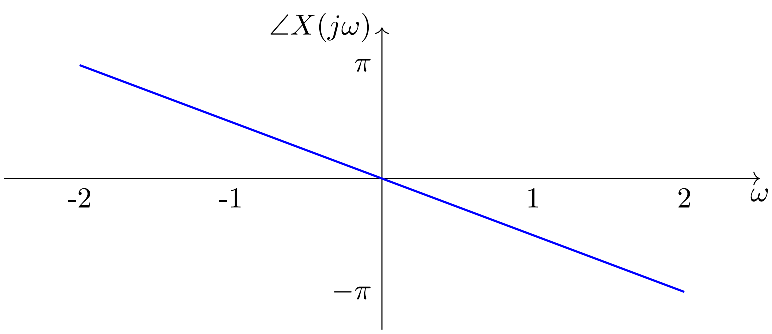
iii. Find $x_s(t)$ in the case where $\frac{2\pi}{T} = \omega_0$.

Solution:
Since $X_s(j\omega)$ covers all the frequency components, $x_s(t)$ is $\delta(t)$.

2. **Fourier transform** (24 points)

(a) (12 points) A signal x(t) has has the following the Fourier Transform.



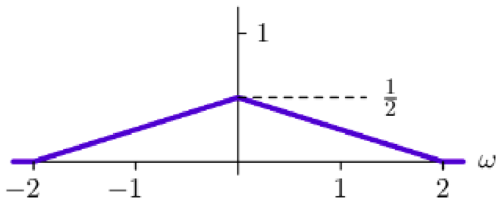


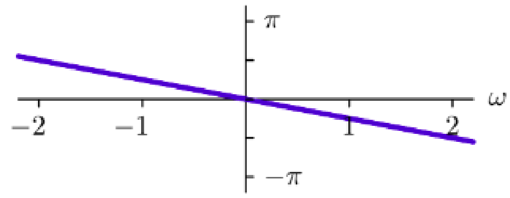
Plot the magnitude and phase plots for the Fourier Transform of the following signals:

i. $x(2t)$

Solution:

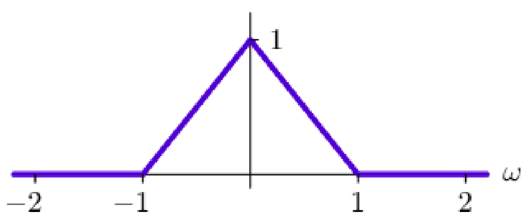
Recall the scaling property of a Fourier transform:
$x(2t) \rightarrow \frac{1}{2}X(\omega/2)$

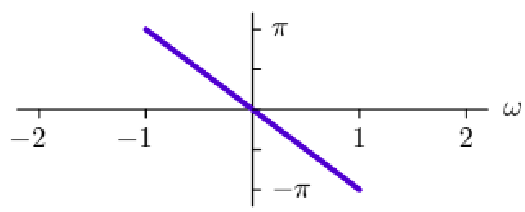




ii. $x(t - \frac{\pi}{2})$

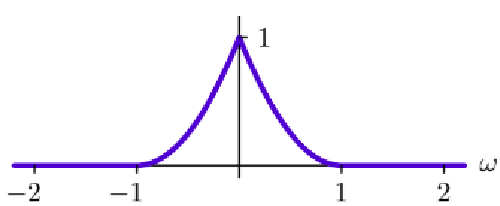
Solution:
Using the shifting property of the Fourier Transform:
$x(t - \frac{\pi}{2}) \rightarrow e^{-j\frac{\pi}{2}\omega}X(j\omega)$
Thus amplitude stays the same, which phase gets altered:
$\angle X_{new}(j\omega) = \angle X(j\omega) - \frac{\pi}{2}\omega = 2\angle X(j\omega)$

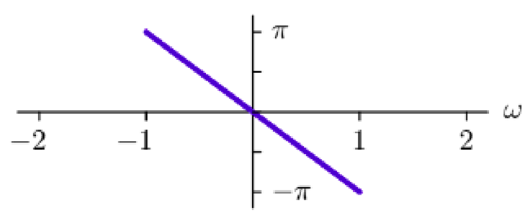




iii. $x(t) * x(t)$

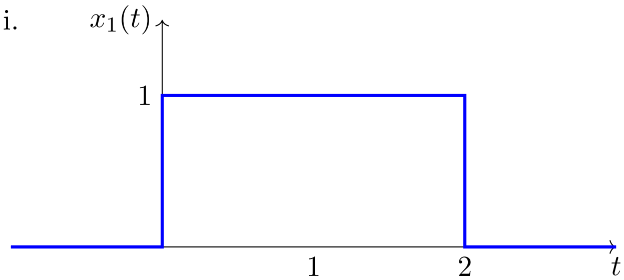
Solution:
Convolution in the time domain is equivalent to multiplication in the Fourier Domain. This results in the plot below
Phase can be obtained similar to part (ii), yielding $\angle X_{new}(j\omega) = 2\angle X(j\omega)$





(b) (12 points) Find the expression of the Fourier transform $X_i(j\omega)$ for the following $x_i(t)$:

i.



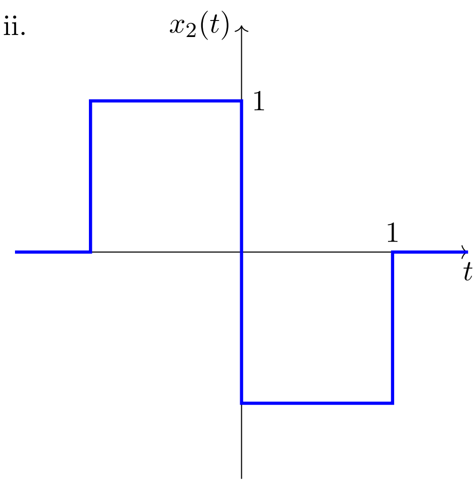
Solution:
Let $x_{1c}(t)$ be a shifted version of $x_1(t)$ so that

$$x_{1c}(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

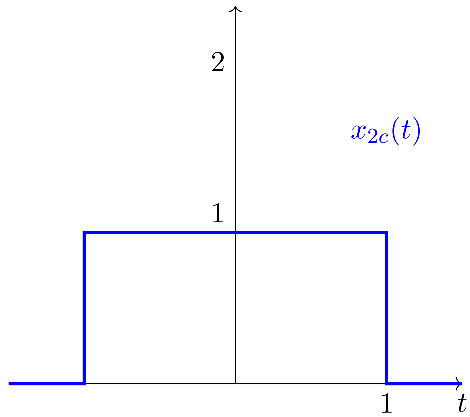
Therefore, $X_{1c}(j\omega) = \frac{2sin(\omega)}{\omega}$. Since $x_1(t) = x_{1c}(t-1)$,

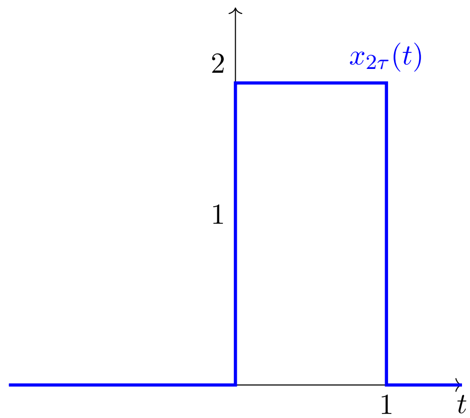
$$\begin{aligned} X_1(j\omega) &= e^{-j\omega} X_{1c}(j\omega) \\ &= 2e^{-j\omega} \frac{sin\omega}{\omega}. \end{aligned}$$

ii.



Solution:
The signal $x_2(t)$ can be expressed as $x_{2c}(t) - x_{2\tau}(t)$ where $x_{2c}(t)$ and $x_{2\tau}(t)$ are shown below.





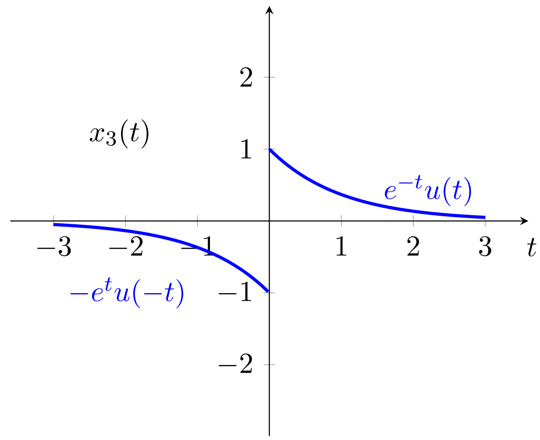
The transform for $x_{2c}(t)$ is $X_{2c}(j\omega) = \frac{sin\omega}{\omega}$. Using the shifting property of Fourier transform, we can easily get $X_{2\tau}(j\omega) = 2e^{-j\frac{\omega}{2}}2\frac{sin\frac{\omega}{2}}{\omega}$. We can get the final expression by combing the previous two transforms:

$$\begin{aligned}
X_2(j\omega) &= X_{2c}(j\omega) - X_{2r}(j\omega) \\
&= \frac{2sin(\omega)}{\omega} - 2e^{-j\frac{\omega}{2}}2\frac{sin\frac{\omega}{2}}{w} \\
&= \frac{2sin(\omega)}{\omega} - \frac{4}{\omega}(cos\frac{\omega}{2} - jsin\frac{\omega}{2})sin\frac{\omega}{2} \\
&= \frac{2sin(\omega)}{\omega} - \frac{2}{\omega}(2cos\frac{\omega}{2}sin\frac{\omega}{2}) + \frac{4}{\omega}jsin^2(\frac{\omega}{2}) \\
&= \frac{2sin\omega}{\omega} - \frac{2sin\omega}{\omega} + \frac{4}{\omega}jsin^2(\frac{\omega}{2})
\end{aligned}$$

$$= j\frac{4}{\omega}sin^2(\frac{\omega}{2}).$$

We can also utilize the 'even and odd' property to check the answer; since $x_2(t)$ is real and odd, $X_2(j\omega)$ is imaginary and odd as expected.

iii.

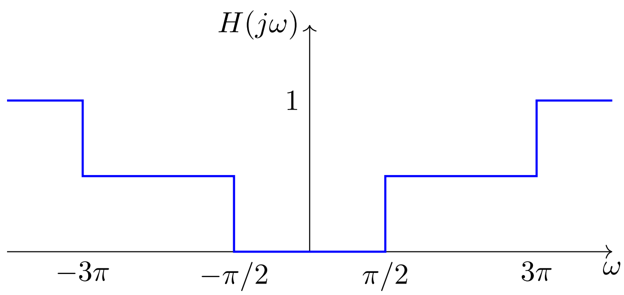


Solution:
Let $w(t) = e^{-t}u(t)$. Then, $x_3(t) = w(t) - w(-t)$ and

$$
\begin{aligned}
X_3(j\omega) &= W(j\omega) - W(-j\omega) \\
&= \frac{1}{1+j\omega} - \frac{1}{1-j\omega} \\
&= -j\frac{2\omega}{1+\omega^2}
\end{aligned}
$$

Since $x_3(t)$ is real and odd, $X_3(t)$ is imaginary and odd as expected.

3. **LTI system** (10 points)
We have the following signal $x(t) = sin(\pi t) + cos^2(2\pi t)$. The $x(t)$ is input the into the LTI which is described by $H(j\omega)$ as shown below.



(a) (3 points) Determine the period of $x(t)$.

Solution:
It should be $2\pi/\pi = 2$. $cos^2(2\pi t)$ will have a period of $1/2$, which is a multiple of 2.

(b) (3 points) Determine the Fourier series coefficients, c_k of $x(t)$.

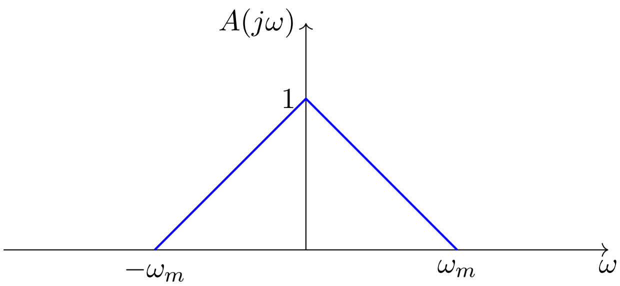
Solution:
Since $x(t) = 1/2 + \frac{e^{j}}{2j} - \frac{e^{-j\pi t}}{2j} + \frac{e^{j4\pi t}}{4} + \frac{e^{-j4\pi t}}{4}$, then $c_0 = 1/2$, $c_1 = c_{-1}^* = \frac{1}{2j}, c_4 = c_{-4} = 1/4$, $c_k = 0$, otherwise.

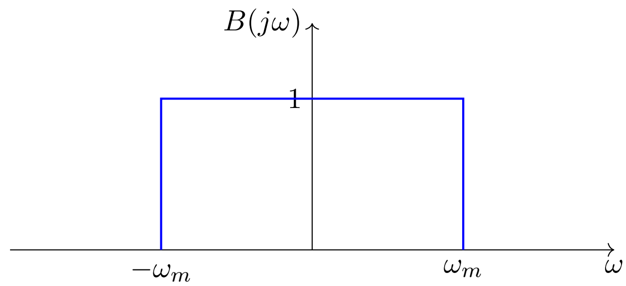
(c) (4 points) Determine the output of $y(t)$.

Solution:
We should consider the eigen function property. Since it is LTI, $y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega_0 t}$. Note that $H(j0) = 0, H(j\pi) = H(-j\pi) = 1$, and $H(j4\pi) = H(-j4\pi) = 2$, then

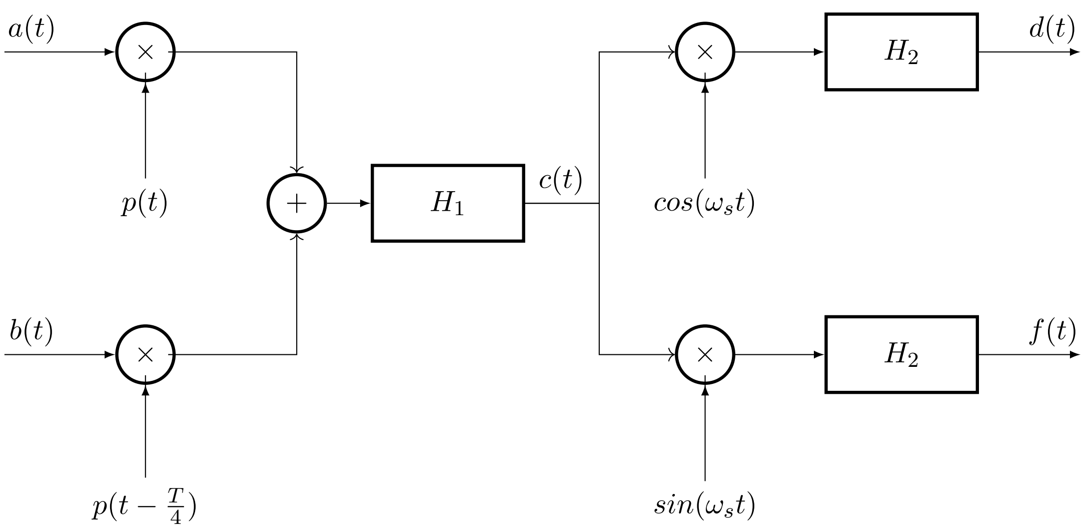
$$y(t) = \frac{e^{j\pi t}}{2j} - \frac{e^{-j\pi t}}{2j} + \frac{e^{j4\pi t}}{2} + \frac{e^{-j4\pi t}}{2} = sin(\pi t) + cos(4\pi t)$$

4. **Modulation and demodulation** (15 points)
It is given that input signals a(t) and b(t) are real and even, with the Fourier Transforms shown below.





Consider the system below:



Where we define the following filters:

$$H_1(j\omega) = \begin{cases} T, & \omega_s - \omega_m \le |\omega| \le \omega_s + \omega_m \\ 0, & otherwise. \end{cases}$$

$$H_2(j\omega) = \begin{cases} 1, & |\omega| \le \omega_m \\ 0, & |\omega| > otherwise. \end{cases}$$

$$p(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$\frac{2\pi}{T} = \omega_s > 2\omega_m$$

Note: You may not need to consider the amplitude scaling in this question.

(a) (4 points) In class, we found the Fourier Transform of the impulse train using the Fourier Series. Using a similar argument, derive the Fourier Transform of the shifted impulse train, $p(t - \frac{T}{4})$.

Solution:

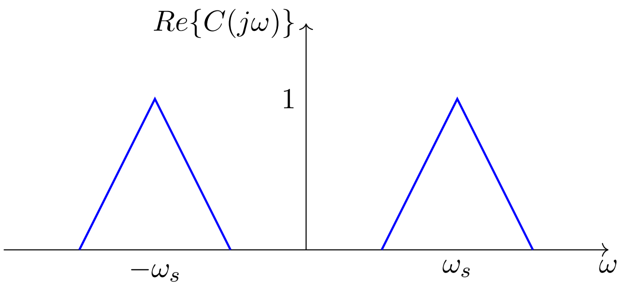
$$\mathcal{F}(\delta_T(t - \frac{T}{4})) = \frac{1}{T}\sum_k \mathcal{F}[e^{j\frac{2\pi}{T}k(t-\frac{T}{4})}] =$$

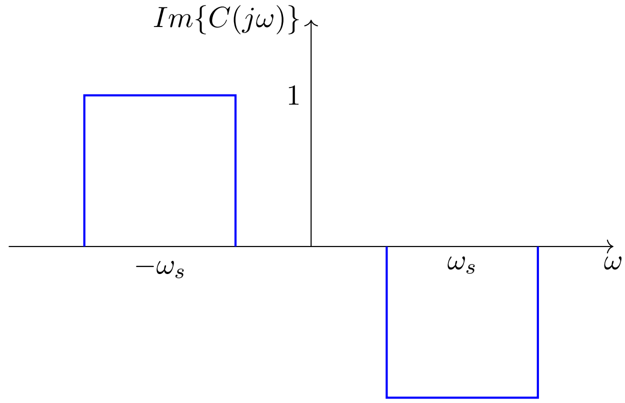
$$\frac{1}{T}\sum_k e^{-j\frac{2\pi k}{4}}\mathcal{F}[e^{j\frac{2\pi}{T}k}] = \frac{1}{T}\sum_k e^{-j\frac{\pi}{2}k}2\pi\delta(\omega - k\omega_0) = \frac{1}{T}\sum_k j^{-k}2\pi\delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_k j^{-k}\delta(\omega - k\omega_0)$$

(b) (4 points) Plot the Real and Imaginary components of $C(j\omega)$ (the Fourier Transform of the output signal of the bandpass filter H_1).

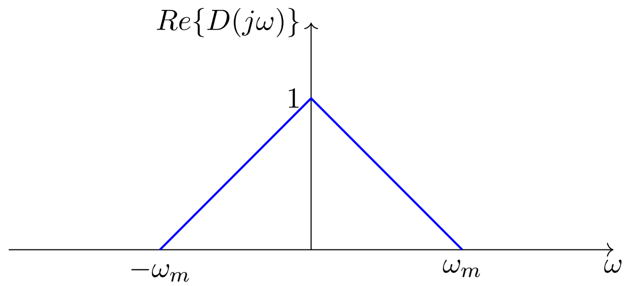
Solution:

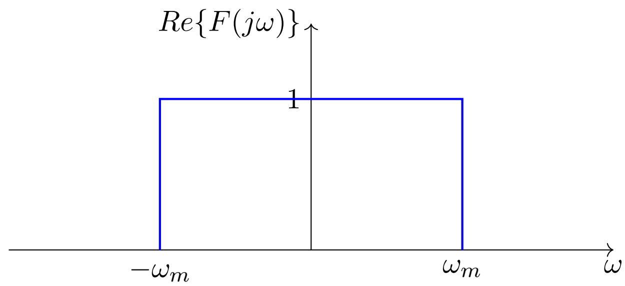




(c) (4 points) Plot the Real and Imaginary Components of $D(j\omega)$ and $F(j\omega)$

Solution:
Both $D(j\omega)$ and $F(j\omega)$ will be Real as Imaginary part is cancelled.

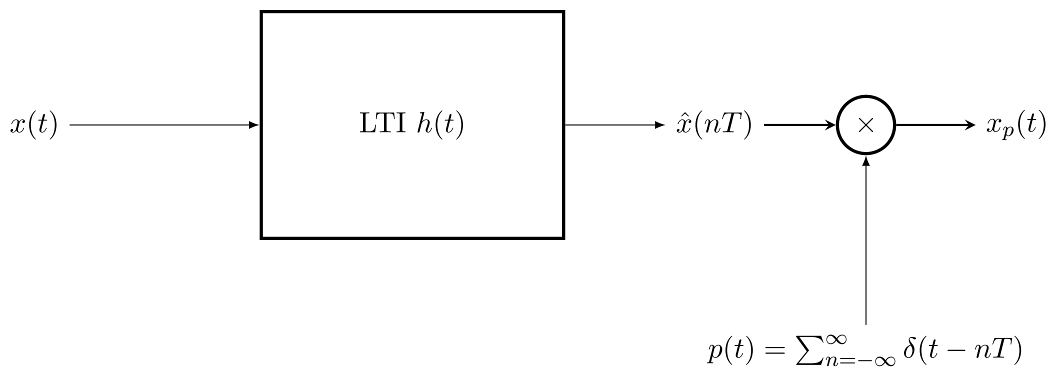




(d) (3 points) Describes d(t) in terms of either a(t) or b(t)

Solution: $d(t)$ is scaled version of $a(t)$

5. **Non-ideal sampling** (15 points)
For the ideal sampling system, the area under the impulse at $t = nT$ would be $x(nT)$. In this non-ideal sampling system the output is an impulse train $x_p(t)$ where the area under the impulse at $t = nT$ is the average value of the input $x(t)$ on the interval $nT - \Delta \leq t \leq nT + \Delta$. The non-ideal sampling system is further illustrated in the following diagram and equations:



$$x_p(t) = \sum_{n=-\infty}^{n=\infty} \hat{x}(nT)\delta(t - nT)$$

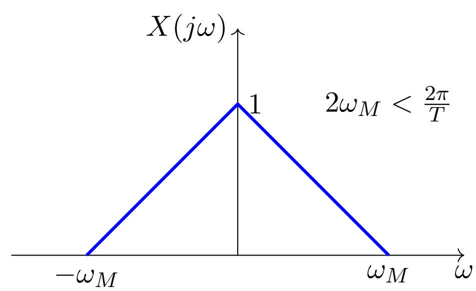
$$\hat{x}(nT) = \frac{1}{2\Delta}\int_{t=nT-\Delta}^{nT+\Delta} x(\tau)d\tau$$

(a) Determine $h(t)$.

Solution:
We find that integration over a symmetric range as convolution with a box over the range, so we have

$$h(t) = \begin{cases} \frac{1}{2\Delta}, & |t| \leq \Delta \\ 0, & |t| > \Delta. \end{cases}$$

(b) Suppose $X(j\omega)$, the Fourier transform of the band-limited input $x(t)$ is as shown below.



Express $X_p(j\omega)$, the Fourier transform of $x_p(t)$, in terms of $X(j\omega)$.

Solution:
Let $Y(j\omega)$ be the output of the filter $h(t)$. $H(j\omega)$, the Fourier transform of $h(t)$, is given by

$$H(j\omega) = \frac{1}{2\Delta}(\frac{2sin(\Delta\omega)}{\omega}) = \frac{sin(\Delta\omega)}{\Delta\omega}$$

$Y(j\omega)$ is then just sampled by the impulse train, we then have

$$\begin{aligned} X_p j\omega &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y(j(\omega - k\frac{2\pi}{T})) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\frac{2\pi}{T}))H(j(\omega - k\frac{2\pi}{T})) \end{aligned}$$

Note that since $\omega_M < \frac{\pi}{T}$, this is just $X(j\omega)H(j\omega)$ made periodic with period $\frac{2pi}{T}$.

(c) Find the maximum value for Δ, such that no frequency component of $x(t)$ is lost due to the non-ideal sampling process.

Solution:
To avoid losing frequency content, we must not multiply $X(j\omega)$ by zero in the range $|\omega| < \omega_M$. We see that the first zero of $H(j\omega)$ occurs at

$$\begin{aligned} sin(\Delta\omega) &= 0 \\ \Delta\omega &= \pi \\ \omega &= \frac{\pi}{\Delta} \end{aligned}$$

so in order to avoid losing frequency content, we need to have

$$\begin{aligned} \omega_M &\leq \frac{\pi}{\Delta} \\ \Delta &\leq \frac{\pi}{\omega_M} \end{aligned}$$

Thus we conclude the largest allowable Δ is $\Delta = \frac{\pi}{\omega_M}$.

6. **Laplace transform** (15 points)

A casual LTI system can be described by the following differential equation:

$$y''(t) + 2y'(t) + y(t) = 2x(t) + 3x''(t).$$

You may assume resting initial conditions (y(0)= 0, y'(0) =0, y"(0) = 0)

(a) Find the transfer function $H(s)$.

Solution:
Applying the Laplace transformation onto the above equation, we get:

$$Y(s)(s^2 + 2s + 1) = X(s)(2 + 3s^2).$$

Hence we can get

$$H(s) = \frac{2 + 3s^2}{s^2 + 2s + 1}.$$

(b) What is the impulse response h(t) of this system?

Solution:

$$\begin{aligned} H(s) &= \frac{(2+3s^2)}{s^2+2s+1} \\ &= 3 - \frac{6}{1+s} + \frac{5}{(1+s)^2}. \end{aligned}$$

Then we can get the impulse response $h(t) = 3\delta(t) - 6e^{-t}u(t) + 5te^{-t}u(t)$.

(c) Given that $\int_{-\infty}^{\infty} x(t)dt = 2.5$, compute $Y(s)$ at $s = 0$.

Solution:
$H(0) = 2$ and $X(0) = \int_{-\infty}^{\infty} x(t)e^{-0t}dt = 2.5$.
Since $Y(s) = X(s)H(s)$, $Y(0) = X(0)H(0) = 5$.

Bonus question (6 points)

(a) (sound) You work at a company that has flown in a famous pianist by the name of Bruinhoven who will play a high-pitched version of the UCLA Fight song. The lowest note that the score asks for is 1000 Hz. In the digital recording the song sounds very low pitched, and you hear off-key notes that are in the hundreds of Hertz - much less than the lowest note in the score. Your boss believes Bruinhoven just had a bad day and played poorly. However, when you were listening in the room to Bruinhoven, the song sounded perfect. When you analyze the recording you also find something curious - these off-key notes are at unusual frequencies that do not map to the specific keys of the piano. What do you think is the most likely explanation, and what is your evidence?

Solution:
Aliasing occurs when a signal contains frequencies which are higher than half of the sampling rate. When we sample these higher frequencies are interpreted as lower frequencies that are not actually present in the signal (the higher frequencies wrap around the sampling rate). This could result in notes which are not actually present in the signal. Evidence would be that you heard the signal properly, but it sounds differently after digital recording. One way to fix this would be to use an ADC with a higher sampling rate.

(b) (light) Can you rediscover the intuition behind a Nobel Prize in Physics? Lasers are like superpowered lightbulbs and tools for humanity. We can shoot lasers far into space or use powerful lasers to perform eye surgery. Unfortunately, lasers do not come in many colors. In the early days of its invention, lasers used to be available at infrared frequencies (topping out at 250 THz). This was not as useful since humans can only perceive frequencies of light from 400 THz to 700 THz. Look up the frequency range of what we perceive as "green". Given the infrared laser tech you have on hand and the tools from this class, how might you generate a green laser?

Solution:
Green light is the 540-580 THz region. We know that a linear method will not be sufficient for generating a frequency higher than one that is present in the input signal. We can accomplish this with a non-linear interaction between the light and a crystal,

which combines the energy of two or more photons into one photon with a harmonic frequency. In this case we can use a Third Harmonic Generating crystal like BBO to generate Green Light within our frequency range.

(c) (light) In this question, you will play with a simple while very effective trick (or philosophy) in signal and systems. The application scenario of this trick is photoplethysmogram (PPG). A PPG is an optically obtained plethysmogram that can be used to detect blood volume changes in the microvascular bed of tissue. A PPG is often obtained by using a pulse oximeter which illuminates visible light onto the skin and measures changes in light absorption using a camera. The blood volume changes can be further utilized for determining the vital signals like heartbeat rate. In industry, some companies try to use the blinking LED instead of steady-state LED as illumination source for PPG. How do you think it might help improve the performance of PPG (i.e., what is the trick behind it)?

Solution:
As we see the keyword 'visible light' in the question, we would have an intuition that the ambient light would be a issue. The ambient light mixes with the PPG signal and strong ambient inevitably deteriorate the accuracy of heartbeat rate measurement.

The blinking LED is a way to enhance the accuracy. We can control the 'on and off' of the LED, when the LED is on, the signal we get from the camera (sensor) is the combination of ambient light and the PPG signal. When the LED is off, we can have only the ambient light. The subtraction of these two enable you to get purified PPG signal.

Note: The philosophy behind this question is that when you want to separate two signals, modulate one of the signal so as to create the difference between these two signals. The 'on and off' of the LED is a way of modulation. In signal and systems, many times you will face cases similar as the PPG example in the above question and you can apply such philosophy to help you boost the performance or 'see the invisible'.