UCLA Department of Electrical Engineering
EE101A – Engineering Electromagnetics
Fall 2016
Midterm, November 1 2016, (1:45 minutes)

Name ______________________ Student number______________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

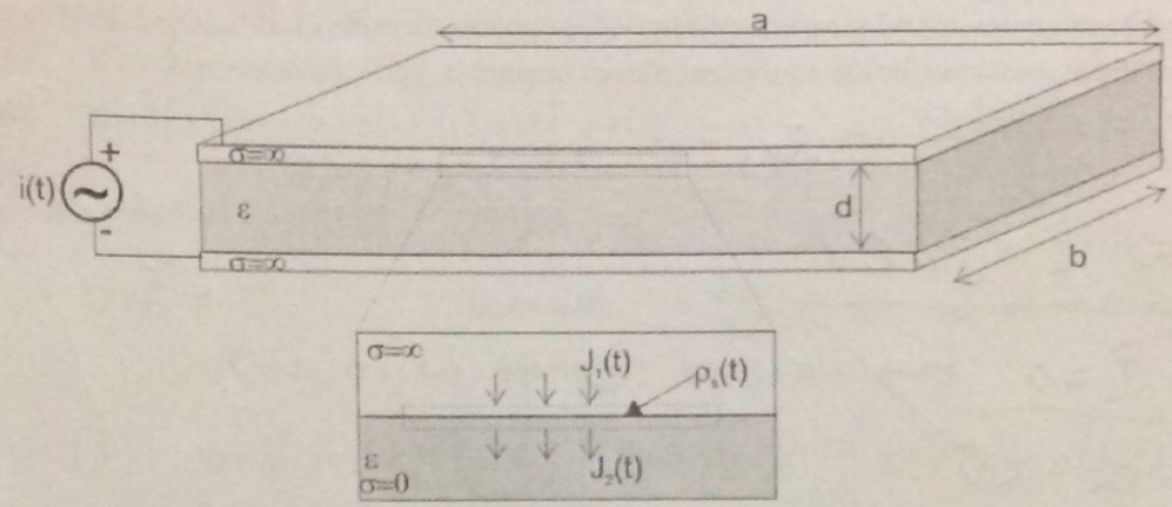
Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	45	45
Problem 2	Electrostatics	10	10
Problem 3	Inductance	45	45
Total		100	100

1. Capacitor (45 points)



(a) (15 points) Consider a parallel plate capacitor with plate dimensions *a* and *b*, and a perfectly insulating dielectric of thickness *d*, and a permittivity *ε*. Assume that the capacitor is hooked up to a current source with time varying current $i(t)=i_0\cos(\omega t)$. The voltage difference between the plates is $v(t)$. The capacitor obeys the standard relation $i(t) = C\, dv(t)/dt$.

Consider a closed Gaussian surface that surrounds the interface between the top metal plate and the dielectric. Write an expression for the current density flowing to the top surface $J_1(t)$, out of the bottom surface $J_2(t)$, and the charge density at the interface $\rho_s(t)$ (all as shown in figure).

① Because the dielectric is perfectly insulating, $\sigma = 0$.
Therefore, no current flows in the dielectric.
i.e. $J_2(t) = 0$

② Assume the charge on the upper plate at time $t=0$ is 0

$\therefore Q = \int_0^t i(t)dt = \frac{i_0}{\omega}\sin(\omega t)$

$\therefore \rho_s(t) = \frac{Q(t)}{ab} = \frac{i_0}{\omega ab}\sin(\omega t)$

③ $\nabla \cdot J = -\frac{\partial \rho}{\partial t} \Rightarrow \int J \cdot dS = -\frac{\partial}{\partial t}\int \rho\, dv$

$\Rightarrow JS = -\frac{\partial}{\partial t}(\rho_s S)$

$\Rightarrow J_1(t) = \frac{\partial}{\partial t}\rho_s = \frac{i_0}{ab}\cos(\omega t)$

(b) (15 points) Now we will use phasors to re-solve (a). Assume the current can be written as $i(t) = Re\{\tilde{I}e^{j\omega t}\}$ and $v(t) = Re\{\tilde{V}e^{j\omega t}\}$ and similarly for all other time varying quantities.
Rewrite the capacitor relation $i(t) = C\, dv(t)/dt$, in phasor form for $\tilde{I}, \tilde{V}$.
Give the expressions for the equivalent phasor quantities: $\tilde{J}_1, \tilde{J}_2, \tilde{\rho}_s$ in terms $\tilde{I}$.

① $i(t) = C\frac{dv(t)}{dt} \Rightarrow Re\{\tilde{I}e^{j\omega t}\} = C\frac{d}{dt}Re\{\tilde{V}e^{j\omega t}\}$

$\Rightarrow \underline{\tilde{I} = j\omega C\tilde{V}}$

② 1. $\underline{\tilde{J}_2 = 0}$ from the same argument as in (a)

2. $\hat{Q} = C\tilde{V}$

$\underline{\tilde{\rho}_s = \frac{\hat{Q}}{ab} = \frac{C\tilde{V}}{ab} = \frac{1}{abj\omega}\tilde{I}}$

3. $J_1 = \frac{d}{dt}\rho_s$

$\Rightarrow \underline{\tilde{J}_1 = j\omega\tilde{\rho}_s = \frac{1}{ab}\tilde{I}}$

(c) (15 points) Challenge problem: Now consider that the dielectric is "leaky", i.e. it has a non-zero conductivity σ. Write new expressions for $\tilde{J}_1, \tilde{J}_2, \tilde{\rho}_s$.

① Because the body of the capacitor can not accumulate charges, the current into the capacitor is the current out.

$\therefore \underline{\tilde{J}_1 = \frac{1}{ab}\tilde{I}}$

② By boundary condition of electric field.
$E = \rho_s/\epsilon$ at the surface of dielectric

$\therefore J_2 = \sigma E = \frac{\sigma\rho_s}{\epsilon}$

Then, by $\nabla\cdot J = -\frac{\partial\rho}{\partial t}$, get

$J_1 - J_2 = \frac{\partial}{\partial t}\rho_s$

$\therefore \tilde{J}_1 - \tilde{J}_2 = j\omega\tilde{\rho}_s$

$\Rightarrow \tilde{J}_1 - \frac{\sigma}{\epsilon}\tilde{\rho}_s = j\omega\tilde{\rho}_s$

$\underline{\tilde{\rho}_s = \frac{\tilde{J}_1}{\frac{\sigma}{\epsilon}+j\omega} = \frac{1}{\frac{\sigma}{\epsilon}+j\omega}\frac{\tilde{I}}{ab}}$

$\therefore \underline{\tilde{J}_2 = \frac{\sigma}{\epsilon}\tilde{\rho}_s = \frac{\sigma}{\sigma+j\omega\epsilon}\frac{\tilde{I}}{ab}}$

2. Electrostatics (10 points)

One of these is an impossible electrostatic field. Which one? You must explain why for credit.

(A): $\mathbf{E} = 4\left[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}\right]$

(B): $\mathbf{E} = 2\left[y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right]$

An electrostatic field obeys

$\nabla \times \vec{E} = 0$, because $-\frac{\partial B}{\partial t} = 0$ in electrostatic case

- For (A) $\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = 4\left[(-2y)\hat{x} + (-3z)\hat{y} + (-x)\hat{z}\right] \neq 0$

$\therefore$ (A) is an impossible electrostatic field 10

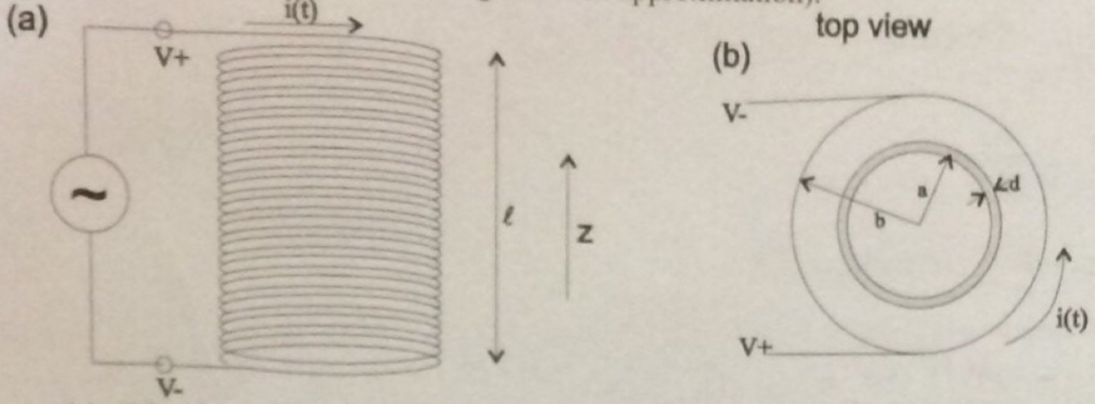
For (B) $\nabla \times \vec{E} = 2\left((2z - 2z)\hat{x} + (2y - 2y)\hat{z}\right)$

$= 0$

$\therefore$ (B) is a possible electrostatic field

3. Inductance (45 points)

Consider a long solenoid of length l, with N turns, and a radius of b as shown in part (a) of the figure. You may consider $l \gg b$ (i.e. the long solenoid approximation).



(a) (15 points) What is the self-inductance of a long solenoid (shown in part (a) of the figure), in terms of fundamental constants, and the parameters mentioned above?

$$L = \frac{\Lambda}{I} = \frac{N\Phi}{I}$$

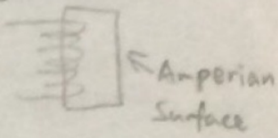
$\nabla \times \vec{H} = \vec{J}$ for magnetostatic case

$\therefore \int \nabla \times \vec{H}\, d\vec{a} = \int \vec{J}\, d\vec{a}$

$\therefore \oint_c \vec{H} \cdot d\vec{l} = IN$

$\therefore Hl = IN$

$\therefore B = \mu \frac{NI}{l} \hat{z}$



$$\Phi = \int \vec{B} \cdot d\vec{s}$$

$\vec{B} = \mu J \hat{z} = \mu \frac{NI}{l} \hat{z}$ for long solenoid inside

$$\therefore \Phi = \mu \frac{NI}{l} \cdot \pi b^2$$

$$\therefore L = \frac{N\Phi}{I} = \frac{\mu N^2 I \pi b^2}{l I} = \frac{\mu N^2 \pi b^2}{l}$$

(b) (15 points) Assume a current $i(t)=I_0 \cos(\omega t)$ is flowing as shown. What is the voltage $v(t)=V_+ - V_-$ at the terminals as a function of time (as shown in part (a)? Pay attention to the sign.

With the directions shown in figure,

$$V = L\frac{di}{dt}$$

$$= -LI_0\omega \sin(\omega t)$$

$$= -\frac{\mu N^2 \pi b^2}{l} I_0 \omega \sin(\omega t)$$

Now imagine a piece of superconducting pipe that is a perfect electrical conductor with radius $a=b/2$, and thickness d is inserted into the center of the solenoid, as shown in part (b) of the figure.

(c) (15 points) Will the addition of the perfectly conducting ($\sigma = \infty$) pipe increase, decrease, or leave unchanged the apparent self-inductance L of the solenoid? Give a qualitative explanation why.

The perfectly conducting pipe will decrease the self-inductance L of the solenoid.

Because the perfect conductor will induce current in itself to cancel the magnetic field within it $(r<a)$ ↑

So the total flux Φ decreases, and hence $L = \frac{N\Phi}{I}$ decreases.

$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial B}{\partial t} \\ \vec{J} = \sigma \vec{E} \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{cases}$$