SOLUTIONS

UCLA Department of Electrical Engineering
EE101 – Engineering Electromagnetics
Winter 2011
Midterm, February 7 2011, (1:45 minutes)

Name ______________________ Student number______________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

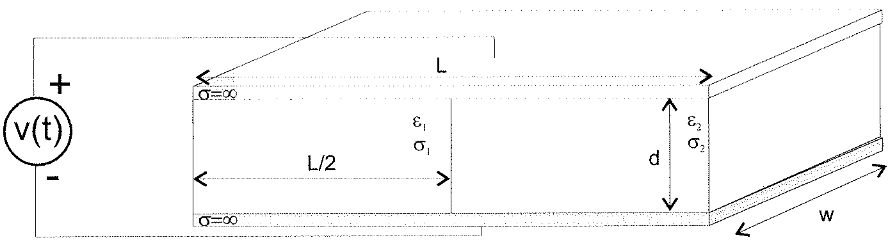
Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	35	
Problem 2	Kirchoff's Laws	30	
Problem 3	Solenoid	35	
Total		100	

Note: Because of photo copying error, Problems 1(b) + 2(b) are missing. Every one received effective full credit for the missing questions.
1b) → 5pts
2(b) → 15 pts

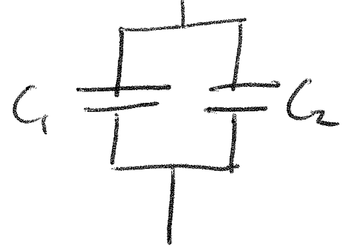
1. Capacitor (~~40~~ 35 points)

In this problem, we consider a parallel plate capacitor shown below where two metal (assume perfectly conducting) plates are separated by distance *d*. Assume that the lateral dimensions L,*b* ≫ *d* so that we can make a 1D approximation and neglect fringing fields. The capacitor is filled with two types of dielectric materials, with permittivity ε_1, and ε_2 assume $\varepsilon_2 > \varepsilon_1$. Because the dielectric material isn't perfect, the material has some small conductivity σ_1, and σ_2 respectively.



(a) (~~10~~ 15 points) If you neglect the conductivity, what is the capacitance of this capacitor, in terms of the dimensions and material properties?

Equivalent to two capacitors in parallel.

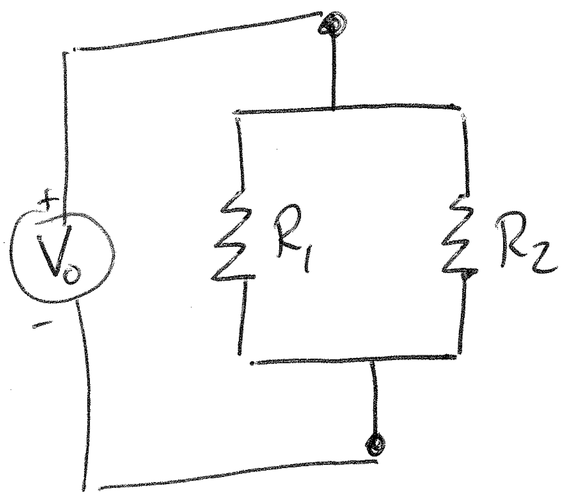


$$C_1 = \frac{WL\varepsilon_1}{2d} \qquad C_2 = \frac{WL\varepsilon_2}{2d}$$

$$C_{total} = C_1 + C_2 = \frac{LW}{2d}(\varepsilon_1 + \varepsilon_2)$$

(c) (15 points) Now consider the leakage conductance of the dielectric – give an expression for the resistance R of this capacitor in terms of the material properties and the dimensions.

Consider two resistors in parallel.



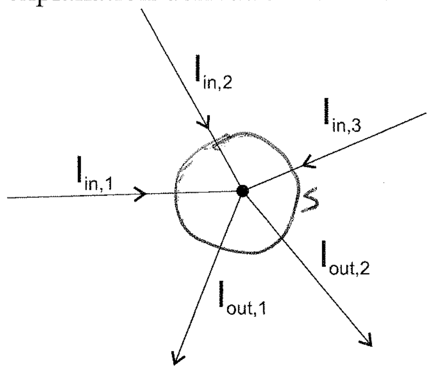
$$R_1 = \frac{2d}{\sigma_1 LW} \qquad R_2 = \frac{2d}{\sigma_2 LW}$$

$$\boxed{R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{2d}{LW}\frac{1}{\sigma_1 + \sigma_2}}$$

2. Kirchoff's Laws (30 points)

(15 pts)

(a) In circuit theory, Kirchoff's current law says that the sum of the currents flowing into a circuit node is equal to the sum of the currents flowing out: $\sum_n I_{in,n} = \sum_m I_{out,m}$. Give a physical explanation/derivation for this rule, citing equations as needed.



Ampere's Law

$$\nabla\cdot(\nabla\times H) = \nabla\cdot J + \nabla\cdot\frac{\partial D}{\partial t}$$

$$\boxed{\nabla\cdot J = -\frac{\partial\rho}{\partial t}}$$ Current continuity

Apply the current continuity equation $\nabla\cdot J = -\frac{\partial\rho}{\partial t}$ in integral form with a surface around the node:

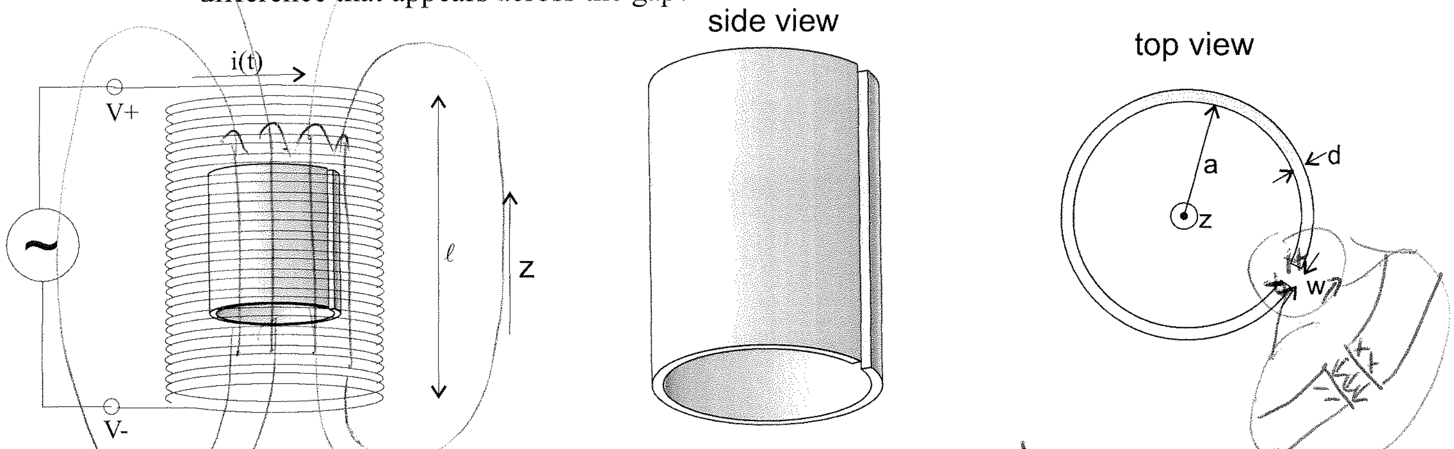
$$\oint \vec{J}\cdot d\vec{S} = -\frac{d}{dt}Q_{enc}$$

In steady state $\frac{dQ}{dt} = 0$ since charge cannot build up in a circuit node in dc (otherwise Coulomb repulsion will create fields to keep current flowing).

Therefore $\oint \vec{J}\cdot d\vec{S} = 0$ in steady state,

which means total current in = total current out at any point in a circuit.

3. Solenoid (~~30~~ 35 points)
   a. (~~15~~ 20 points) Consider a cylindrical metal shell with conductivity σ placed inside a long solenoid of length l with N turns driven to produce a magnetic field. The shell has a small gap in the edge as shown (width $w \ll a$). The current in the solenoid is $i(t)=kt$, where k is a positive constant (units of A/s), with a direction as indicated in the figure. The shell has dimensions as shown, where $d \ll a$.
   What is the current density $\mathbf{J}(\phi,t)$ flowing in the metal shell ? What is the potential difference that appears across the gap?



Field generated by solenoid inside: $\vec{H} = \frac{i(t)N}{\ell}\hat{z} = \frac{ktN}{\ell}\hat{z}$

$$\frac{d\vec{H}}{dt} = z\frac{kN}{\ell} \quad \left(\begin{array}{l}\text{this creates}\\ \text{a steady emf}\end{array}\right)$$

This will induce an emf around circumference of metal shell.

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \oint\vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int\vec{B}\cdot d\vec{S} = -\frac{d}{dt}\Phi$$

Choose loop through circumference of metal.

$$\frac{d}{dt}\Phi = -\pi a^2 \mu_0 \frac{kN}{\ell} = \oint\vec{E}\cdot d\vec{l} = E_\phi 2\pi a$$

See next page for alternate sol'n.

$$E_\phi = \frac{-\mu_0 kN}{\ell}\frac{a}{2}$$

$\boxed{J=0}$ because gap won't let ~~current flow in steady state.~~

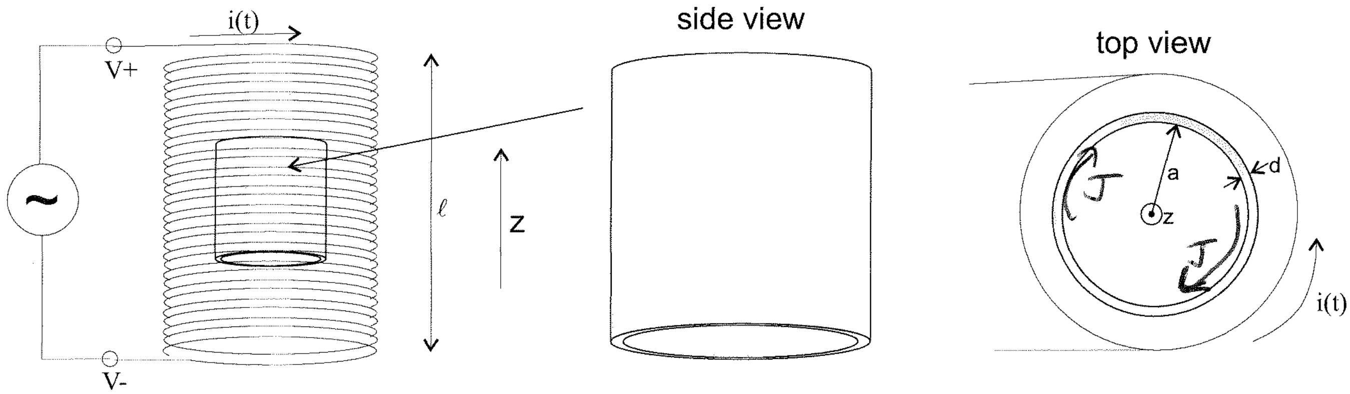
Potential across gap $= \oint d\phi r\, E_\phi = 2\pi a E_\phi = \frac{-\pi a^2 \mu_0 kN}{\ell}$

b. (15 points) Now consider if the metal shell has no gap as shown. Once again, $i(t)=kt$. What is the current density $\mathbf{J}(\phi,t)$ flowing in the metal shell, and which direction does it flow? Your answer should be as a function of dimensional parameters, k, etc. - H should not appear in your final answer.

You may wish to refer to the relationship we derived in class for a metal shell in a z-oriented H-field:

$$\frac{\partial}{\partial t}H_{in} + \frac{H_{in}}{\tau_m} = \frac{H_0}{\tau_m},$$ where $\tau_m = d\sigma a\mu_0/2$ is the magnetic diffusion time, and where

$H_1(t)$ is the field generated by currents in the metal shell, $H_0(t)$ is the externally applied H field in the z-direction resulting from the solenoid, and $H_{in}(t)=H_1(t)+H_0(t)$ is the total axial H-field inside the metal shell (all H-fields are defined to be oriented in the z-direction).



The steady increase in $i(t)$ creates a steady increase in $H_0(t) = \frac{ktN}{\ell}$

$\frac{d}{dt}H_0 = \frac{kN}{\ell}$ ⇒ This will create a steady emf which drives a steady current density $\boxed{J_\phi = \sigma E_\phi = -\frac{\mu_0 kN}{\ell}\frac{a}{2}\sigma\,\hat{\phi}}$

While this J will create its own magnetic field H_1, since J_ϕ is steady $\left(\frac{dJ}{dt}=0\right)$, $\frac{dH_1}{dt}=0$, + only $\frac{dH_0}{dt} = \frac{dH_{in}}{dt}$ will contribute to the emf in the shell.

See next page for transient response

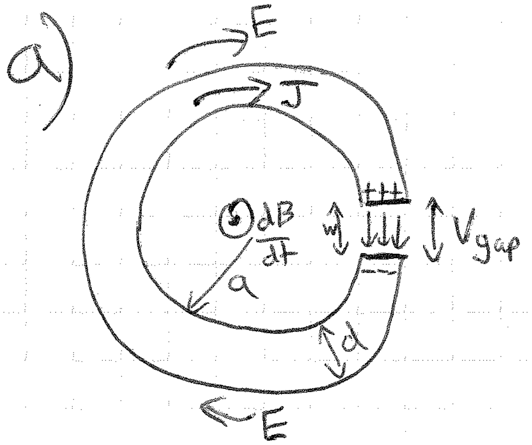
Problem 3 Addendum - Transient Case (not required)

While the problem specified a steadily increasing solenoid current $i(t) = kt$, some of you interpreted this as

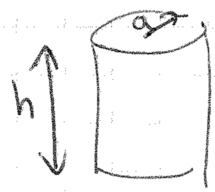
$$i(t) = \begin{cases} 0 & t<0 \\ kt & t>0 \end{cases}$$

This will cause a turn-on transient in the response.

a)

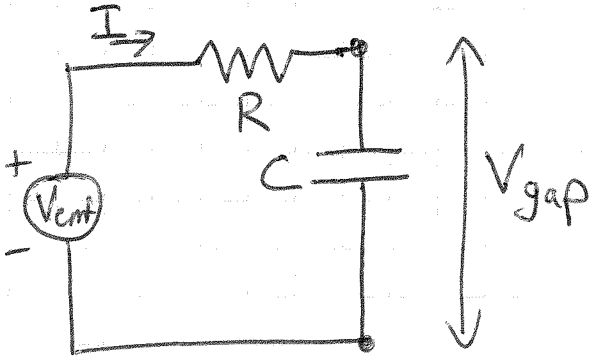


After an initial turn on of current, the gap will act as a capacitor to produce an opposite field that counteracts the emf field, & stops the current from flowing.



h = height of metal cylinder

Equivalent Circuit



R is resistance of conductor

$$R = \frac{2\pi a}{\sigma d h}$$ (assume $w \ll a$)

$$C \approx \frac{\varepsilon_0 h d}{w}$$ (assume $w \ll d$ Parallel plate capacitor approx)

$$\boxed{RC = \frac{2\pi a}{w}\frac{\varepsilon_0}{\sigma}}$$ Dielectric relaxation time for this circuit

When $i(t)$ is turned on at $t=0$

$$V_{emf} = -\frac{d}{dt}\Phi$$

$$2\pi a E_\phi = -\frac{\pi a^2 \mu_0 k N}{l}$$

a) continued

Inhomogeneous ODE

$$I = \frac{V_{emf} - V_{gap}}{R} = \frac{C\,dV_{gap}}{dt} \Rightarrow \boxed{V_{emf} = V_{gap} + RC\frac{dV_{gap}}{dt}}$$

Sol'n to ODE:

$$V_{gap} = V_{emf}\left(1 - e^{-t/RC}\right)$$

Initial cond: $V_{gap}(t=0) = 0$
Final cond: $V_{gap}(t\to\infty) = V_{emf}$

$$\boxed{V_{gap} = \frac{\pi a^2 \mu_0 k N}{\ell}\left(1 - e^{-t/RC}\right)}$$

as $t \to \infty$ V_{gap} reaches steady constant value: $\frac{\pi a^2 \mu_0 k N}{\ell}$

$$\boxed{J = \frac{I}{dh} = -\hat{\phi}\,\frac{\sigma a \mu_0 k N}{2\ell}\,e^{-t/RC}}$$

as $t \to \infty$, $J \to 0$, which makes sense.

Steady current can't flow across a gap.

The process takes several RC times to settle down.

For a good conductor like Copper $\frac{\varepsilon_0}{\sigma} \sim 10^{-19}$ s

Very fast!

Current only flows for a short time!

b) <u>No gap</u> $\quad i(t) = \begin{cases} 0 & t<0 \\ kt & t>0 \end{cases}$

Now we can use the differential eq we derived in class for a metal pipe in an axial magnetic field.

$$H_0 = ktN/l$$
$$H_{in} = H_0 + H_1$$

$$\frac{\partial}{\partial t} H_{in} + \frac{H_{in}}{\tau_m} = \frac{H_0}{\tau_m}$$

$$H_1 = J_s = J\,d \quad \text{(in metal shell)}$$

$$\tau_m = \frac{\sigma \mu_0 d a}{2}$$

$$\frac{\partial}{\partial t} H_0 + \frac{\partial}{\partial t} H_1 + \frac{H_1}{\tau_m} + \cancel{\frac{H_0}{\tau_m}} = \cancel{\frac{H_0}{\tau_m}}$$

$$\boxed{\frac{d}{dt} H_1 + \frac{H_1}{\tau_m} = -\frac{d}{dt} H_0}$$

Solve inhomogeneous ODE

Homogeneous soln: $\quad H_{1h} = Ae^{-t/\tau_m}$

Particular soln: $\quad H_{1p} = B\frac{d}{dt}H_0 = B\frac{kN}{l}$

Full soln: $H_1(t) = Ae^{-t/\tau_m} + B\frac{kN}{l}$ $\qquad A, B = \text{coeff}$

Init cond: $H_1(t=0) = 0$ $\qquad$ Final cond $t \to 0$ steady state

$$H_1(t \to \infty) = -\frac{\tau_m kN}{l}$$

$B = -\tau_m \quad A = \tau_m kN/l$

$$H_1(t) = -\frac{\tau_m kN}{l}\left(1 - e^{-t/\tau_m}\right)$$

$$H_{in} = H_0 + H_1 = \frac{\tau_m kN}{l}\left(\frac{t}{\tau_m} - 1 + e^{-t/\tau_m}\right)$$

$$\boxed{\vec{J}(t) = -\hat{\phi}\,\frac{\tau_m kN}{dl}\left(1 - e^{-t/\tau_m}\right) = -\hat{\phi}\,\frac{\sigma a \mu_0 kN}{2l}\left(1 - e^{-t/\tau_m}\right)}$$

After several τ_m $(t \to \infty)$ J goes to a steady value

$$\vec{J} = -\hat{\phi}\,\frac{\tau_m kN}{dl} = -\hat{\phi}\,\frac{\sigma a \mu_0 kN}{2l}$$