[CS M51A W18] Solutions for Midterm exam

Date: 02/12/18

Problem 1 (20 points)

Digit Value	BCD 8421	2421	Excess-3	2-out-of-5
0	0000	0000	0011	00011
1	0001	0001	0100	11000
2	0010	0010	0101	10100
3	0011	0011	0110	01100
4	0100	0100	0111	10010
5	0101	1011	1000	01010
6	0110	1100	1001	00110
7	0111	1101	1010	10001
8	1000	1110	1011	01001
9	1001	1111	1100	00101

1. **(4 points)** What are the decimal numbers, x and y, represented in the specified code?

$$x = (110000111011)_{\texttt{2421}}$$
$$y = (110000111011)_{\texttt{EXCESS-3}}$$

Solution
Look up the corresponding bit patterns from the table above. $x = 635$, $y = 908$

2. **(5 points)** When the length of a bit vector is 20, how many different integers can be represented by using

   (a) BCD code
   (b) 2421 code
   (c) Excess-3 code
   (d) 2-out-of-5 code
   (e) Binary

Solution
(a) (b) (c): Each decimal digit takes 4 bits. 10^5
(d): Now each decimal digit takes 5 bits. 10^4.
(e): $2^{20} = 1024 \times 1024 = 1048576 \approx 10^6$

3. **(3 points)** Is the 2421 code is unique? If not, list all digit values that have more than one 2421 code and show codes different from 2421 code shown in the above table for these digit values.

   Solution
   $d = d_3 \times 2 + d_2 \times 4 + d_1 \times 2 + d_0 \times 1$
   From this equation, we know that d_3 and d_1 are interchangeable or d_2 can absorb 1s from d_3 and d_1. For example, either $\underline{1}1\underline{0}0$ or $\underline{0}1\underline{1}0$ can be used for a digit 6. For a digit 3, either $\underline{0}0\underline{1}1$ or $\underline{1}0\underline{0}1$ can be used. For a digit 5, either $\underline{1}0\underline{1}1$ or $0\underline{1}01$ can be used. For each $2 <= x <= 7$, there are two representations.

4. **(5 points)** In general, how many different 2421 codes are there? Explain briefly.

   Solution
   There are 2^6 different codes.

5. **(3 points)** (`True`/`False`) Are all 2421 codes are self-complementing? If not, how many self-complementing codes are there?

   Solution
   False. 2^3 codes exist. While $2 <= d <= 4$ has two representations, the 9's complement, $5 <= 9 - d <= 7$, is determined by d.

Problem 2 (20 points)

Design a combinational system that has two inputs x and y in $\{0, 1, 2, 3\}$ represented in the binary code. The output is the product $p = (x \times (3y + 1)) \bmod 8$ also in the binary code.

1. **(4 points)** Show a high-level description of the system.

   Solution
   The input set (for the variable x and y) $= \{0, 1, 2, 3\}$
   The output set (we use z for the output) $= \{0, 1, 2, 3, 4, 5, 6, 7\}$
   The function $z = p = (x \times (3y + 1)) \bmod 8$.

   or the table may be drawn.

y \ x	0	1	2	3
0	0	1	2	3
1	0	4	0	4
2	0	7	6	5
3	0	2	4	6

2. **(4 points)** Show a binary-level description of the system as truth tables of the output switching functions z_2, z_1, z_0.

   Solution

y_1y_0 \ x_1x_0	00	01	10	11
00	000	001	010	011
01	000	100	000	100
10	000	111	110	101
11	000	010	100	110

3. **(4 points)** Derive a sum of minterms in m-notation for the output z_2.

   Solution
   $z_2(x_1, x_0, y_1, y_0) = x_1'x_0y_1'y_0 + x_1'x_0y_1y_0' + x_1x_0'y_1y_0' + x_1x_0'y_1y_0 + x_1x_0y_1'y_0 + x_1x_0y_1y_0' + x_1x_0y_1y_0$
   $= m(5, 6, 10, 11, 13, 14, 15)$

4. **(4 points)** Derive a minimal sum of products (SOP) for the output z_2.

   Solution
   $z_2 = x_0 y_1' y_0 + x_0 y_1 y_0' + x_1 y_1$

5. **(4 points)** Draw a NAND-NAND network that implements the minimal SOP for z_2. Assume that the complements are available.

   Solution
   $z_2 = \text{NAND}(\text{NAND}(x_0, y_1', y_0), \text{NAND}(x_0, y_1, y_0'), \text{NAND}(x_1, y_1))$

Problem 3 (20 points)

$a+b=b+a$	$ab=ba$	Commutativity
$a+(bc)=(a+b)(a+c)$	$a(b+c)=(ab)+(ac)$	Distributivity
$a+(b+c)=(a+b)+c=a+b+c$	$a(bc)=(ab)c=abc$	Associativity
$a+a=a$	$aa=a$	Idempotency
$a+a'=1$	$aa'=0$	Complement
$1+a=1$	$0a=0$	
$0+a=a$	$1a=a$	Identity
$(a')'=a$		Involution
$a+ab=a$	$a(a+b)=a$	Absorption
$a+a'b=a+b$	$a(a'+b)=ab$	Simplification
$(a+b)'=a'b'$	$(ab)'=a'+b'$	DeMorgan's law

For the function $E(w,x,y,z)$, we are given the following boolean expression

$$E(w,x,y,z)=w'x'y'z+w'xy'z'+w'xy'z+w'xyz'+wx'y'z$$

1. (4 points) Simplify the given boolean expression. Indicate at each simplification step the identity used.

Solution:
Method 1:

$$\begin{aligned}
E(w,x,y,z) &= w'x'y'z+w'xy'z'+w'xy'z+w'xyz'+wx'y'z \\
&= w'xy'z'+w'xy'z+w'xyz'+wx'y'z+w'x'y'z && \text{...Commutativity} \\
&= w'xy'z'+w'xy'z+w'xyz'+(w+w')x'y'z && \text{...Distributivity} \\
&= w'xy'z'+w'xy'z+w'xyz'+(1)x'y'z && \text{...Complement} \\
&= w'xy'z'+w'xy'z+w'xyz'+x'y'z && \text{...Identity} \\
&= w'xy'(z'+z)+w'xyz'+x'y'z && \text{...Distributivity} \\
&= w'xy'(1)+w'xyz'+x'y'z && \text{...Complement} \\
&= w'xy'+w'xyz'+x'y'z && \text{...Identity} \\
&= w'x(y'+yz')+x'y'z && \text{...Distributivity} \\
&= w'x(y'+z')+x'y'z && \text{...Simplification} \\
&= w'xy'+w'xz'+x'y'z && \text{...Distributivity}
\end{aligned}$$

2. (4 points) Complete the K-map for the given expression.
Solution:

wx \ yz	00	01	11	10
00	0	1	0	0
01	1	1	0	1
11	0	0	0	0
10	0	1	0	0

3. (4 points) Which of the following are the essential prime implicants for the above expression:

   (a) $w'y'z$ (b) $x'y'z$ (c) $w'xyz'$ (d) $w'xz'$

   Solution: **(b) and (d)**

4. (2 points) Which of the following are the minimal SOP expression for the above function:

   (a) $w'y'z + x'y'z + w'xy' + w'xz'$ (b) $w'y'z + w'xy' + w'xyz' + wx'y'z$

   (c) $w'y'z + x'y'z + w'xz$ (d) $x'y'z + w'xy' + w'xz'$

   Solution: **(d)**

5. (4 points) Which of the following are the essential prime implicates for the above expression:

   (a) $w' + x'$ (b) $w' + y'$ (c) $y' + z'$ (d) $x + z$

   Solution: **(a), (c), (d)**

6. (2 points) Which of the following are the minimal POS expression for the above function:

   (a) $(w' + x').(w' + y').(y' + z').(x + y')$ (b) $(w' + x').(w' + y').(y' + z').(x + z)$

   (c) $(w' + x').(y' + z').(x + z)$ (d) $(w' + x').(y' + z').(x + y')$

   Solution: **(c)**

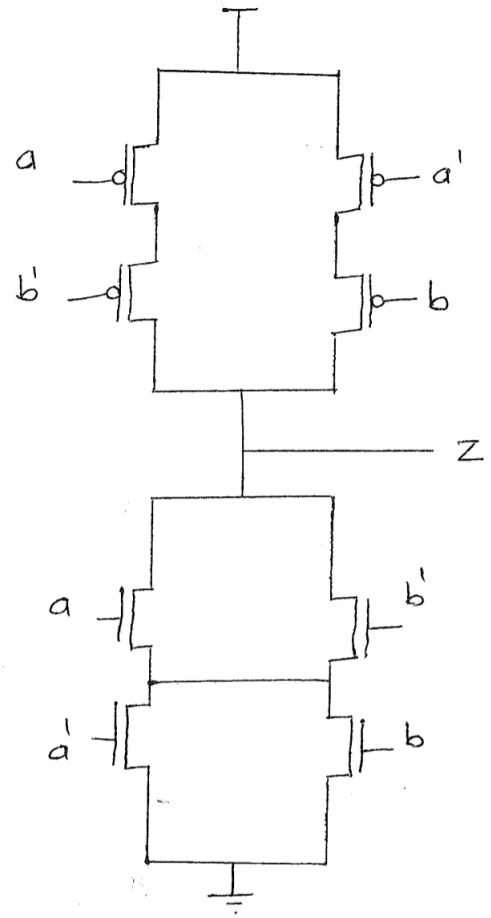
Problem 4 (8 points)

Implement a single complex CMOS gate for the XOR operation.

1. (2 points) Write the expression for the pull-down network.
   Solution:
   $z = a \oplus b$
   i.e. $z = a'b + b'a$
   Hence $z' = (a'b + b'a)' = (a + b').(b + a')$

2. (2 points) Write the expression for the pull-up network.
   Solution:
   $z = a \oplus b$
   i.e. $z = a'b + b'a$

3. (4 points) Draw the CMOS network based on the above expressions.
   Solution:



Or any equivalent CMOS network.

Problem 5 (12 points)

$$E(a, b, c) = ac' + a'bc + ab'c$$

1. **(6 points)** Prove that $E(a, b, c)$ is a universal function. You can use constant 1 or 0 as an input.
   Solution

   E(0,b,c) = bc = AND(b,c)
   E(1,b,c) = b'c+c' = b'+c' = NAND(b,c)
   E(1,1,c) = c' = NOT(c)

   Since {NAND} is a universal set, E is a universal function as well.

2. **(6 points)** Implement the following boolean function using only "E" gates. You can use constant 0 or 1 as inputs.

   $$F(w, x, y, z) = w'x'yz + y'x$$

   Solution
   We can rewrite the F function using two-input NAND and AND

   F(w,x,y,z) = OR (AND(AND(w',x'),AND(y,z)) , AND(y',x))

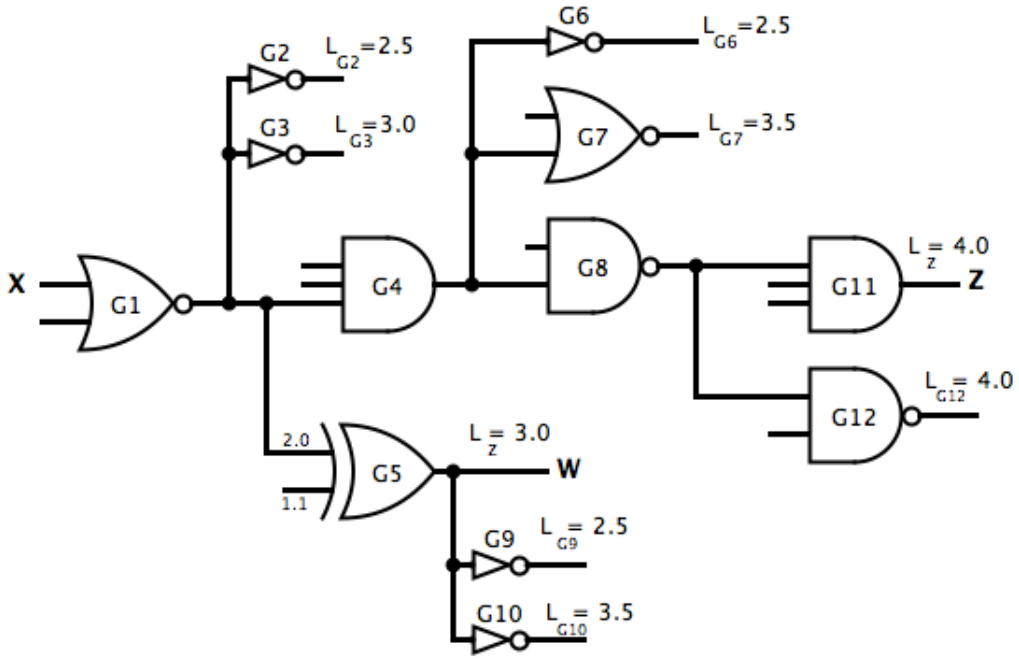
   = NAND(NAND (AND (NOT (w), NOT(x)) , AND (y,z)), AND (NOT(y),x))

   = E(1, E(1, E(0, E(1,1,w), E(1,1,x)), E(0,y,z)), E(0,E(1,1,y),x))

Problem 6 (20 points)

Determine the propagation delay of the gate network shown. The outputs are z and w, and the input signal that we are interested in is x. The related gate characteristics are given in the table below.

Gate Type	Fan-in	Propagation Delays (ns) t_{pLH}	Propagation Delays (ns) t_{pHL}	Load Factor I
NOT	1	$0.02 + 0.038L$	$0.05 + 0.017L$	1.0
NOR	2	$0.06 + 0.075L$	$0.07 + 0.016L$	1.0
NAND	2	$0.05 + 0.038L$	$0.08 + 0.027L$	1.0
AND	3	$0.20 + 0.038L$	$0.18 + 0.018L$	1.0
XOR	2	$0.30 + 0.036L$	$0.30 + 0.021L$	1.1
		$0.16 + 0.036L$	$0.15 + 0.020L$	2.0



1. **(10 points)** Find the worst case value of $t_{pLH}(x \to z)$. Fill in the blanks below with the appropriate values.

   Solution

Gate type & fan-in	G1: NOR2 $\to$ G4: AND3 $\to$ G8: NAND2 $\to$ G11: AND3
LH / HL	G1: HL $\to$ G4: HL $\to$ G8: LH $\to$ G11: LH
Total load L	G1: 5.0 $\to$ G6: 3.0 $\to$ G8: 2.0 $\to$ G11: 4.0

   For the propagational delay values:

   $$\begin{aligned} \text{G1:} &\quad 0.07 + 0.016 \cdot 5.0 = 0.15 \\ \text{G4:} &\quad 0.18 + 0.018 \cdot 3.0 = 0.234 \\ \text{G8:} &\quad 0.05 + 0.038 \cdot 2.0 = 0.126 \\ \text{G11:} &\quad 0.20 + 0.038 \cdot 4.0 = 0.352 \end{aligned}$$

   $t_{pLH}(x \to z) = 0.15 + 0.234 + 0.126 + 0.352 = 0.862$ (ns)

2. **(8 points)** Unlike other gates, a low to high input for an XOR gate can cause the output to transition both from low to high and from high to low, depending on the value of the other input gate.

x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

From the table, when $x = 0$, a low to high ($0 \to 1$) transition at input y will cause the output to move from low to high, but when $x = 1$, the same low to high transition at y will cause the output to move from high to low.

Taking this into consideration, find the worst case value of $t_{pLH}(x \to w)$.

Because G5 is an XOR gate, we need to consider both low to high and high to low transitions at the output of G1, and select the worst case. Fill in the blanks below with the appropriate values.

Solution

Gate type & fan-in	G1: NOR2 $\to$ G5: XOR2
Total load L	G1: 5.0 $\to$ G5: 5.0
LH / HL	G5: LH

For the propagational delay values:

$$\begin{aligned} \text{G5:} &\quad 0.16 + 0.036 \cdot 5.0 = 0.34 \\ \text{G1(LH):} &\quad 0.06 + 0.075 \cdot 5.0 = 0.435 \\ \text{G1(HL):} &\quad 0.07 + 0.016 \cdot 5.0 = 0.15 \end{aligned}$$

3. **(2 points)** What is the worst case value of $t_{pLH}(x \to w)$?

Solution For gate G1, LH has the higher delay. Therefore, the worst case value of $t_{pLH}(x \to w)$ is $0.435 + 0.34 = 0.775$ (ns).