[CS M51A Fall 19] Solutions for Midterm exam

Date: 11/5/19

TAs

1.	$a + b$	$= b + a$	ab	$= ba$	Commutativity
2.	$a + (bc)$	$= (a + b)(a + c)$	$a(b + c)$	$= (ab) + (ac)$	Distributivity
3.	$a + (b + c)$	$= (a + b) + c$	$a(bc)$	$= (ab)c$	Associativity
		$= a + b + c$		$= abc$	
4.	$a + a$	$= a$	aa	$= a$	Idempotency
5.	$a + a'$	$= 1$	aa'	$= 0$	Complement
6.	$1 + a$	$= 1$	$0a$	$= 0$	
7.	$0 + a$	$= a$	$1a$	$= a$	Identity
8.	$(a')'$	$= a$			Involution
9.	$a + ab$	$= a$	$a(a + b)$	$= a$	Absorption
10.	$a + a'b$	$= a + b$	$a(a' + b)$	$= ab$	Simplification
11.	$(a + b)'$	$= a'b'$	$(ab)'$	$= a' + b'$	DeMorgan's Law

Problem 1 (10 points)

The following questions are based on the function described below. This is a minority function, where the output is 1 when less than half of the inputs are 1.

Inputs: $a, b, c \in \{0, 1\}$

Outputs: $z \in \{0, 1\}$

Function: $z = \begin{cases} 1 & \text{if one or zero inputs are 1,} \\ 0 & \text{otherwise} \end{cases}$

1. **(4 points)** Complete the given table.

Solution

a	b	c	$z = E(a, b, c)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

2. **(6 points)** If the constant input 0 is allowed (but not 1), does this function form a universal set? Is it universal if 1 is allowed but not 0? Show proof.

Solution The expression we can get directly from the table is

$$\begin{aligned} E(a,b,c) &= a'b'c' + a'b'c + a'bc' + ab'c' \\ &= a'b'(c' + c) + a'bc' + ab'c' \\ &= a'b' + a'bc' + ab'c' \\ &= a'(b' + bc') + ab'c' \\ &= a'b' + a'c' + ab'c' \\ &= b'(a' + ac') + a'c' \\ &= a'b' + b'c' + a'c' \end{aligned}$$

Fixing the value of c to 0, we can see that

$$E(a, b, 0) = a'b' + a' + b' = a'(b' + 1) + b' = a' + b' = (ab)'$$

This is an implementation of a NAND gate, which is a universal set. Therefore, the minority function forms a universal set when $c = 0$.

Fixing the value of c to 1,

$$E(a, b, 1) = a'b' = (a + b)'$$

which is a NOR. Thus the minority function forms a universal set when $c = 1$.

Problem 2 (10 points)

Given the following simplification of a boolean expression, answer the following.

$$
\begin{aligned}
& (ab' + c')'(b' + c)(a + bc') && (1)\\
= \;& (ab')'c'(b' + c)(a + bc') && (2)\\
= \;& (a' + b)c'(b' + c)(a + bc') && (3)\\
= \;& (a' + b)c'(ab + ac + bb'c' + bcc') && (4)\\
= \;& (a' + b)c'(ab + ac) && (5)\\
= \;& (a' + b)(ab + ac)c' && (6)\\
= \;& (aa'b + abb + aa'c + abc)c' && (7)\\
= \;& abc' + abcc' && (8)\\
= \;& abc' && (9)
\end{aligned}
$$

1. **(4 points)** There is at least one mistake in this simplification. Find **<u>all</u>** steps that are derived incorrectly from its previous step (for example, write (8)→(9) if equation (9) is derived incorrectly from (8)).

   Solution The wrong steps are (1)→(2) (no invert on c') and (3)→(4) (first term on last sum term should be ab').

2. **(6 points)** Show the correct simplification of (1).

   Solution The correct simplification is shown below.

$$
\begin{aligned}
& (ab' + c')'(b' + c)(a + bc')\\
= \;& (ab')'c(b' + c)(a + bc')\\
= \;& (a' + b)c(b' + c)(a + bc')\\
= \;& (a' + b)c(ab' + ac + bb'c' + bcc')\\
= \;& (a' + b)c(ab' + ac)\\
= \;& (a' + b)(ab' + ac)c\\
= \;& (aa'b' + abb' + aa'c + abc)c\\
= \;& abcc = abc
\end{aligned}
$$

Problem 3 (15 points)

Design a two-level gate network of the following system.

Inputs:	$x, y \in \{0, 1, 2, 3\}$
Outputs:	$z \in \{0, 1, 2, 3\}$
Function:	$z = \{3xy + 3\} \bmod 4$

1. **(3 points)** Complete the switching table using binary encoding for all values.

   Solution

x_1	x_0	y_1	y_0	$3xy+3$	z	z_1	z_0
0	0	0	0	3	3	1	1
0	0	0	1	3	3	1	1
0	0	1	0	3	3	1	1
0	0	1	1	3	3	1	1
0	1	0	0	3	3	1	1
0	1	0	1	6	2	1	0
0	1	1	0	9	1	0	1
0	1	1	1	12	0	0	0
1	0	0	0	3	3	1	1
1	0	0	1	9	1	0	1
1	0	1	0	15	3	1	1
1	0	1	1	21	1	0	1
1	1	0	0	3	3	1	1
1	1	0	1	12	0	0	0
1	1	1	0	21	1	0	1
1	1	1	1	30	2	1	0

2. **(4 points)** Show the switching expression of z_1 and z_0 in sum of minterms form.

Solution Looking at the table, we can get:

$$\begin{aligned} z_1 &= \sum m(0,1,2,3,4,5,8,10,12,15) \\ z_0 &= \sum m(0,1,2,3,4,6,8,9,10,11,12,14) \end{aligned}$$

3. **(4 points)** Show the switching expression of z_1 and z_0 in product of maxterms form.

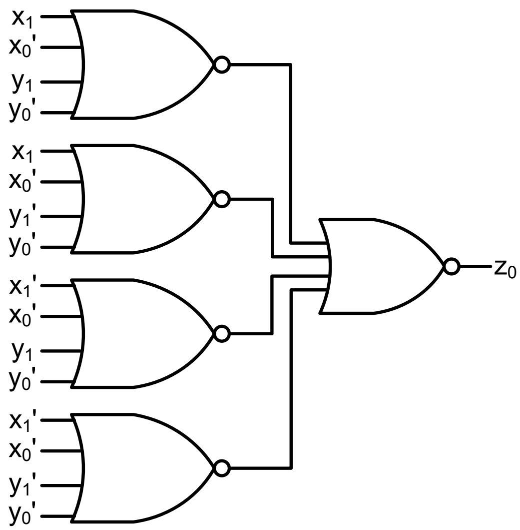
Solution Looking at the table, we can get:

$$\begin{aligned} z_1 &= \prod M(6,7,9,11,13,14) \\ z_0 &= \prod M(5,7,13,15) \end{aligned}$$

4. **(4 points)** Draw the two-level NAND-NAND or NOR-NOR network for the **canonical** sum of products or product of sums form of z_0 (and not z_1). Assume that complemented inputs are available. Select the two-level network that requires the smaller total number of NAND or NOR gates (do not count NOT gates).

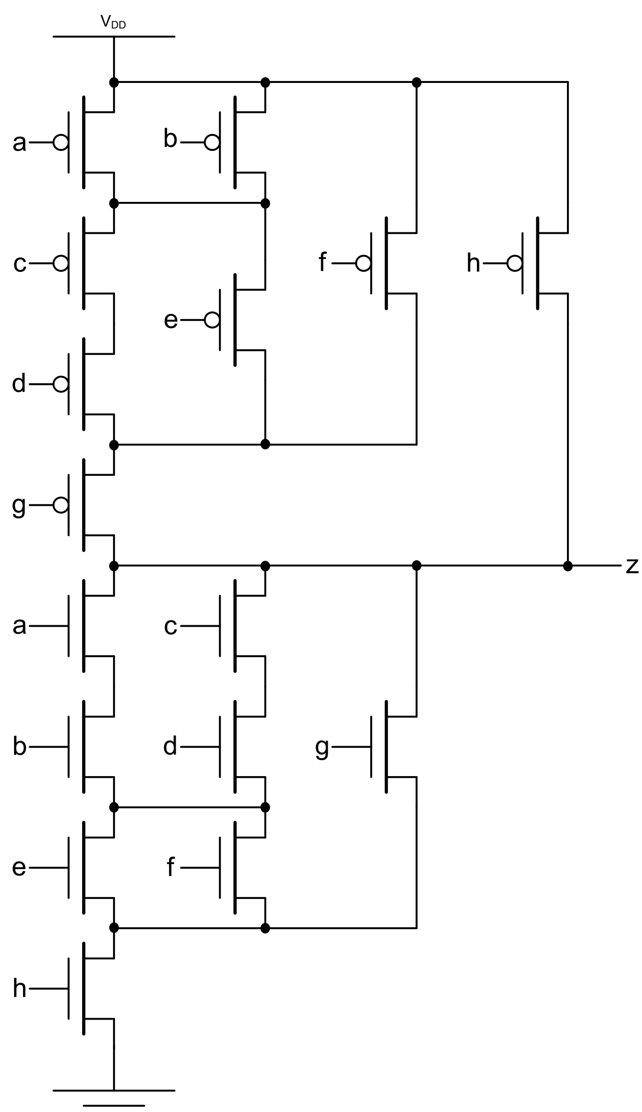
Solution For z_0, we can see that it requires 12 minterms or 4 maxterms. Since we are asked to use the form that gives us the smaller number of gates, we can use the product of maxterms form to build a NOR-NOR two-level network.

$$\begin{aligned} z_0 &= \prod M(5,7,13,15) \\ &= (x_1+x_0'+y_1+y_0')(x_1+x_0'+y_1'+y_0')(x_1'+x_0'+y_1+y_0')(x_1'+x_0'+y_1'+y_0') \\ &= \left[(x_1+x_0'+y_1+y_0')'+(x_1+x_0'+y_1'+y_0')'+(x_1'+x_0'+y_1+y_0')'+(x_1'+x_0'+y_1'+y_0')'\right]' \end{aligned}$$



Problem 4 (15 points)

Answer the following questions about the given CMOS circuit.



1. **(7 points)** There is a problem in this CMOS circuit. When $a = 1$, $f = 1$ and $h = 1$, there exists at least one combination of signals that activates both the pull-up and pull-down networks and forms a direct path from V_{DD} to ground. Show any single combination of values that makes this happen.

   Solution For the pull-up network:

$$z = ((a' + b')(c'd' + e') + f')g' + h'$$

   For the pull-down network:

$$z' = ((ab + cd)(e + f) + g)h$$

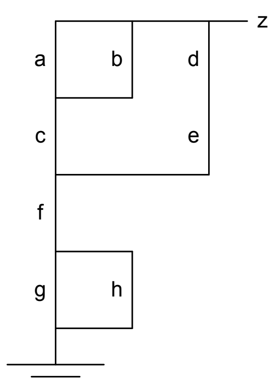
   For a direct path to form from V_{DD} to ground, both networks need to be connected. As $h = 1$, $g = 0$ otherwise the pull-up network will not be active.

   For the pull-up, since $f = 1$, we need a path through $(a'+b')(c'd'+e')$. And because $g = 0$, for the pull-down, we need a path through $(ab + cd)(e + f)$. As $a = 1$, we need $b = 0$ for $(a' + b') = 1$. With $a = 1$ and $b = 0$, we need $c = d = 1$ for $(ab + cd) = 1$. And this in turn implies that $e = 0$ so that $(c'd' + e') = 1$.

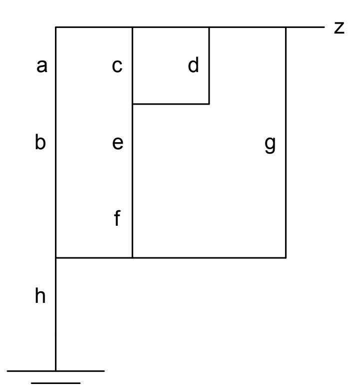
   The resulting signal values are: $b = 0$, $c = 1$, $d = 1$, $e = 0$ and $g = 0$.

2. **(8 points)** Assuming that the pull-up network has the correct functionality that we want, **show the expression** for the corresponding pull-down network. Which one of the following is the expression equivalent to? (each letter in the diagram stands for an NMOS transistor)

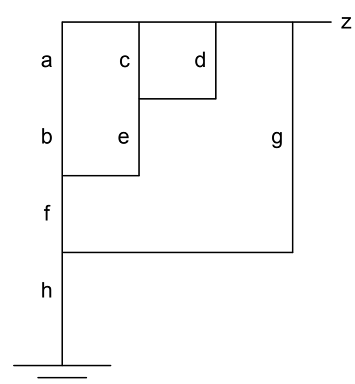
   (a) Pull-down network 1



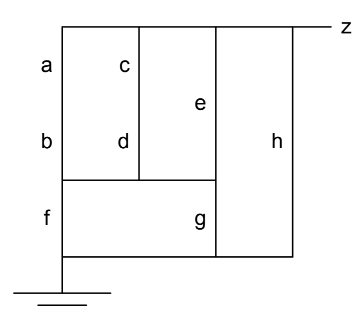
   (b) Pull-down network 2



   (c) Pull-down network 3



   (d) Pull-down network 4



Solution

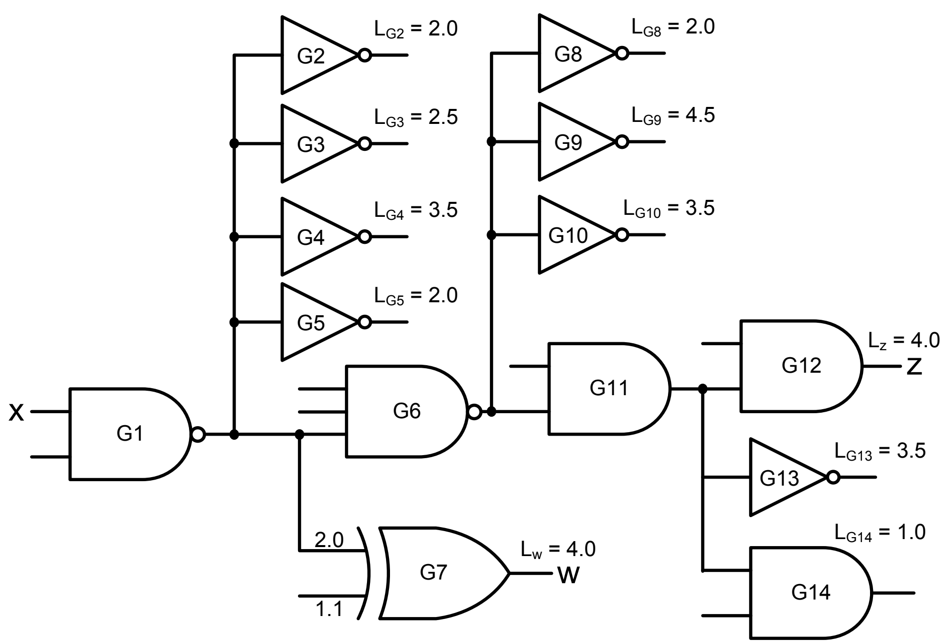
$$
\begin{aligned}
z &= ((a'+b')(c'd'+e')+f')g'+h' \\
z' &= \left[((a'+b')(c'd'+e')+f')g'+h'\right]' \\
&= (((a'+b')(c'd'+e')+f')g')'h \\
&= (((a'+b')(c'd'+e')+f')'+g)h \\
&= (((a'+b')(c'd'+e'))'f+g)h \\
&= (((a'+b')'+(c'd'+e')')f+g)h \\
&= ((ab+(c'd')'e)f+g)h \\
&= ((ab+(c+d)e)f+g)h
\end{aligned}
$$

This is equivalent to (c).

Problem 5 (20 points)

Consider the following gate network. The gate characteristics are given in the table below.

Gate Type	Fan-in	Propagation Delays (ns) t_{pLH}	t_{pHL}	Load Factor I
NOT	1	$0.02+0.038L$	$0.05+0.017L$	1.0
AND	2	$0.15+0.037L$	$0.16+0.017L$	1.0
NAND	2	$0.05+0.038L$	$0.08+0.027L$	1.0
NAND	3	$0.07+0.038L$	$0.09+0.039L$	1.0
XOR	2	$0.30+0.036L$	$0.30+0.021L$	1.1
		$0.16+0.036L$	$0.15+0.020L$	2.0



1. **(8 points)** Determine the propagation delay $t_{pLH}(x \to z)$. Assume that the unconnected inputs to G1, G6, G11 and G12 have value 1 to allow changes in x to propagate to z. Fill in the blanks below with the appropriate values.

Solution

Gate type and fan-in	G1: NAND2 G6: NAND3 G11: AND2 G12: AND2
LH / HL	G1: HL → G6: LH → G11: LH → G12: LH
Output load L	G1: 7.0 G6: 4.0 G11: 3.0 G12: 4.0

The propagational delays of the gates in the path are:

$$\begin{aligned}
\text{G1:}\quad & 0.08 + 0.027L = 0.08 + 0.027 \cdot 7.0 = 0.269 \\
\text{G6:}\quad & 0.07 + 0.038L = 0.07 + 0.038 \cdot 4.0 = 0.222 \\
\text{G11:}\quad & 0.15 + 0.037L = 0.15 + 0.037 \cdot 3.0 = 0.261 \\
\text{G12:}\quad & 0.15 + 0.037L = 0.15 + 0.037 \cdot 4.0 = 0.298
\end{aligned}$$

$t_{pLH}(x \to z) = 0.296 + 0.222 + 0.261 + 0.298 = 1.077$ (ns)

2. **(8 points)** Unlike other gates, a low to high input for an XOR gate can cause the output to transition both from low to high and from high to low, depending on the value of the other input.

x	y	XOR
0	0	0
0	1	1
1	0	1
1	1	0

From the table, when $x = 0$, a low to high $(0 \to 1)$ transition at input y will cause the output to move from low to high, but when $x = 1$, the same low to high transition at y will cause the output to move from high to low.

Taking this into consideration, find the worst case value of $t_{pLH}(x \to w)$.

Because G7 is an XOR gate, we need to consider both low to high and high to low transitions at the output of G1, and select the worst case. Fill in the blanks below with the appropriate values.

Solution

Gate type and fan-in	G1: NAND2 → G7: XOR2
Output load L	G1: 7.0 G7: 4.0
LH / HL	G7: LH

For the propagational delay values:

$$\begin{aligned}
\text{G7:}\quad & 0.16 + 0.036L = 0.16 + 0.036 \cdot 4.0 = 0.304 \\
\text{G1(LH):}\quad & 0.05 + 0.038L = 0.05 + 0.038 \cdot 7.0 = 0.316 \\
\text{G1(HL):}\quad & 0.08 + 0.027L = 0.08 + 0.027 \cdot 7.0 = 0.269
\end{aligned}$$

3. **(4 points)** What is the worst case value of $t_{pLH}(x \to w)$?

Solution For gate G1, LH has the worse delay compared to the HL case. Therefore, the worst case value of $t_{pLH}(x \to w)$ is $0.304 + 0.316 = 0.62$ (ns).

Problem 6 (18 points)

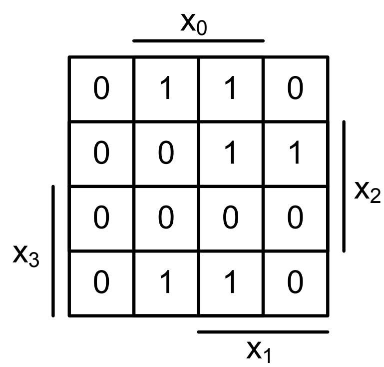
Given $f(x_3, x_2, x_1, x_0) = x_3'x_2'x_1'x_0 + x_3'x_2'x_1x_0 + x_3'x_2x_1x_0' + x_3'x_2x_1x_0 + x_3x_2'x_1'x_0 + x_3x_2'x_1x_0$

1. **(4 points)** Fill out the following K-map.

   Solution From the given equation, we can get

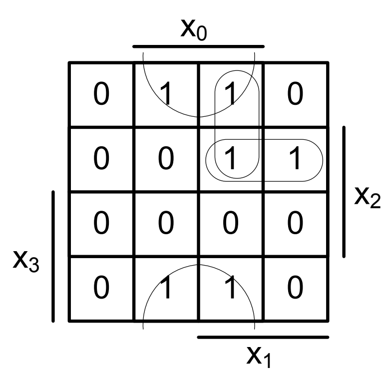
   $$f(x_3, x_2, x_1, x_0) = \sum m(1, 3, 6, 7, 9, 11)$$

   The completed K-map is shown:



2. **(4 points)** Find and circle all the prime implicants. How many implicants are there? Show.

   Solution The prime implicants are shown in the K-map.



   The prime implicants are $x_2'x_0$ (d), $x_3'x_1x_0$ (h) and $x_3'x_2x_1$.

   There are 13 implicants: $m(1, 3, 6, 7, 9, 11)$, $x_2'x_0$, $x_3'x_1x_0$, $x_3'x_2x_1$, $x_3'x_2'x_0$, $x_2'x_1'x_0$, $x_2'x_1x_0$, and $x_3x_2'x_0$.

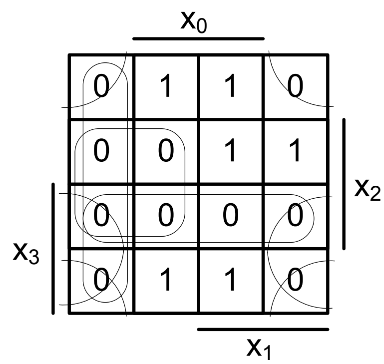
3. **(2 points)** Which are the essential prime implicants? ***Solution*** The 1 rectangles with single coverage are 1, 6, 9 and 11. So we have two essential prime implicants, $x_2'x_0$ and $x_3'x_2x_1$.

4. **(1 point)** Write the minimal sum of products expression for f. Is it unique?

   Solution Since all 1 cells are covered by the essential prime implicants, the minimal SOP expression is $x_2'x_0 + x_3'x_2x_1$. It is unique.

5. **(4 points)** Find all the prime implicates and show them on the K-map.

   Solution The prime implicates are shown in the K-map.



The prime implicates are $(x_3' + x_2')$ (c), $(x_3' + x_0)$ (missing), $(x_2' + x_1)$ (d), $(x_2 + x_0)$ (f) and $(x_1 + x_0)$ (h).

6. **(2 points)** Write down all the essential prime implicates.

   Solution The 0 cells with single coverage are 2, 5 and 11. 2 is covered by $(x_2 + x_0)$, 5 is covered by $(x_2' + x_1)$ and 11 is covered by $(x_3' + x_2')$.

7. **(1 point)** Write the minimal product of sums expression for f. Is it unique?

   Solution Since all 0 squares are covered by the essential prime implicants, the minimal POS is $(x_2+x_0)(x_2'+x_1)(x_3' + x_2')$. It is unique.

Problem 7 (12 points)

Obtain the minimal expressions for the following combinational system using K-maps. The system is a Gray code-to-BCD converter.

Input: $x \in \{0, 1, \ldots, 9\}$ in Gray code

Output: $z \in \{0, 1, \ldots, 9\}$ in BCD

The Gray code table is shown below:

Digit	Gray code		
0	0000	8	1100
1	0001	9	1101
2	0011	10	1111
3	0010	11	1110
4	0110	12	1010
5	0111	13	1011
6	0101	14	1001
7	0100	15	1000

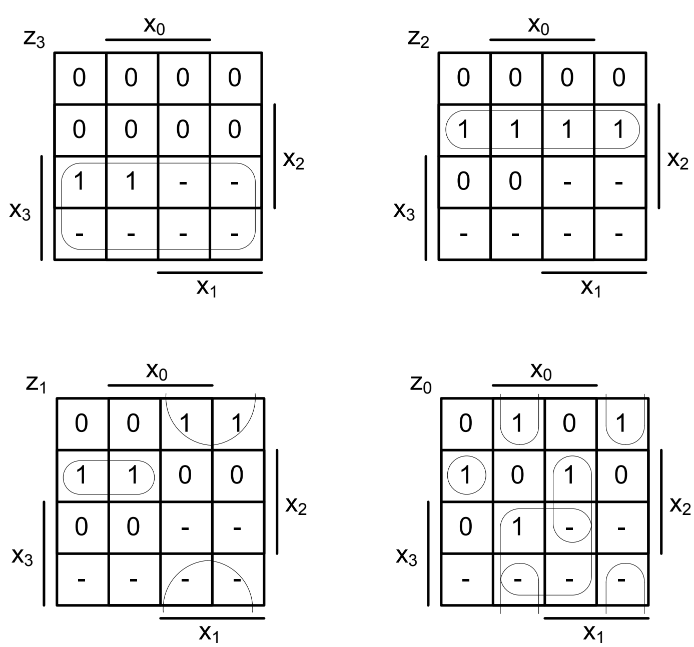
1. **(4 points)** Complete the binary switching table.

   Solution Values 10 to 15 are don't-cares as the input and output set are set to a single digit.

x_3	x_2	x_1	x_0	Value in Gray code	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0	1
0	0	1	0	3	0	0	1	1
0	0	1	1	2	0	0	1	0
0	1	0	0	7	0	1	1	1
0	1	0	1	6	0	1	1	0
0	1	1	0	4	0	1	0	0
0	1	1	1	5	0	1	0	1
1	0	0	0	15	–	–	–	–
1	0	0	1	14	–	–	–	–
1	0	1	0	12	–	–	–	–
1	0	1	1	13	–	–	–	–
1	1	0	0	8	1	0	0	0
1	1	0	1	9	1	0	0	1
1	1	1	0	11	–	–	–	–
1	1	1	1	10	–	–	–	–

2. **(8 points)** Use K-maps to find the **minimal** sum of products expression for each output.

Solution



The minimal SOP expressions are

$$
\begin{aligned}
z_3 &= x_3 \\
z_2 &= x_3{}'x_2 \\
z_1 &= x_3{}'x_2x_1{}' + x_2{}'x_1 \\
z_0 &= x_3{}'x_2x_1{}'x_0{}' + x_2{}'x_1{}'x_0 + x_2{}'x_1x_0{}' + x_2x_1x_0 + x_3x_0
\end{aligned}
$$