Name: Erynn - Marie Phan
Student ID: 504 757 459

CS181 Winter 2019 - Midterm

You will have 110 minutes to take this exam. This exam is open-book and open-notes, but any material not used in this course is prohibited. There are two questions and a total of 150 points and the second question has one part worth 30 extra credit points. You will not get 20% points for just writing "I don't know". However, if you describe a non-trivial approach that you tried using to solve the problem but realized it doesn't work and explain correctly why it doesn't work and then write "I don't know" you will get 20% points for that problem. It is solely the grader's discretion as to whether your stated approach was indeed non-trivial. Place your name and UID on every page of your solutions. Please use separate pages for each question.

Honor Code Agreement: I understand this exam is open-book and open-notes, but any material not used in this course is strictly prohibited. I also understand that this exam is to be taken individually without any outside help (except possibly from the professor or the TAs) within the time limits set forth. I agree to adhere to the course honor code and if I am unsure of any rules of the honor code, I will ask for clarification from the professor or the TAs.

Signature: Erynn-Marie Phan

Question	Points
1	70
2	65
EC	0
Total	135

Note: While solving the problems, you can always reference theorems and claims from your homeworks, lectures, and anything in the book without having to prove those again.

504 757 459

We will prove the following pumping lemma in two parts:

Lemma 1. *For all FDFAs F, there exists an $n \in \mathbb{N}$ such that for all $x \in L(F)$ with $|x| \geq n$, there exist strings a, b, c, d, e such that $x = abcde$, and*

i. $\forall i,\ ab^icd^ie \in L(F)$

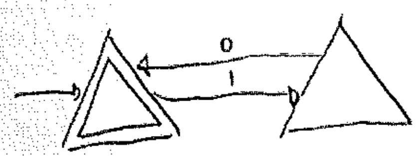
ii. $|b| + |d| > 0$

iii. $|abde| \leq n$

c) **(30 points).** Let an Even-Flip-FDFA F be a FDFA where every cycle in the state diagram of F has an even number of flipping states. Prove that **Lemma 1** holds for all Even-Flip FDFAs. *Hint: Part (1b) should help.*

d) **(20 points).** Prove **Lemma 1** holds for *all* FDFAs.
Note that proving (1d) also proves (1c).

a)



15

b) 03476521

0347 03476521 6521

10

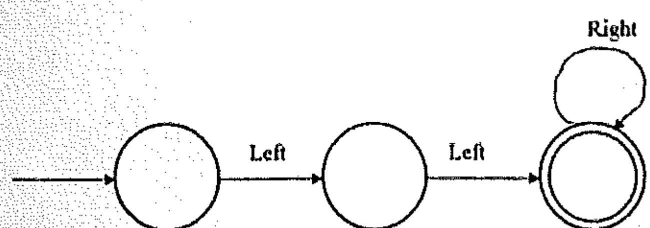
c and d) see pages 4-8

Problem 2. A *Nondeterministic Line Robot (NLR)* is a robot that moves along a number line. It is controlled by a hardwired program which is like an NFA except that each of its transitions require no input, take one time step to carry out, and are accompanied by a direction in $\{Left, Right\}$ indicating the direction that the robot should move on that time step. An NLR is specified solely by this control NFA.

An NLR operates as follows: It is placed at the origin of an infinite number line and powered on, putting it in its start state. At each time step while it is powered on, it must nondeterministically choose to follow one of the transitions from its current state. It will then move one unit in the direction indicated by the label on the transition, either $Left$ or *Right*. After moving, if it reaches an accepting state from this transition, then it may nondeterministically choose to power down. Alternatively, it can choose not to power down and to continue operation. If there is a case where the robot cannot transition or power down, then it self-destructs and explodes.

The location-set of an NLR R, denoted by $Loc(R)$, is the set of all positions on the number line where the robot R might end up when it powers down. Note that self-destructing and exploding at a position does not count as powering down at that position. Formally, $Loc(R) = \{x \mid R \text{ could power down at position } x\}$.

Example: The location-set generated by the NLR specified below is $\{x \mid x \geq -2\}$:



a) **(15 points).** Given two fixed numbers $a, b \in \mathbb{N}$, specify a NLR for the location-set $L = \{ax + by \mid x, y \in \mathbb{Z}\}$.

b) **(30 points).** We will prove the following theorem in two parts

Theorem 1. *Let an Unproductive-NLR be a NLR R such that every cycle in the state diagram of R has an equal number of right and left transitions. If R is an Unproductive-NLR, then Loc(R) is finite.*

i. **(20 points).** Prove that if R is an Unproductive-NLR and there is a path through R that directs R to position k in m time steps, then there exists some path to position k with no repeated states.

ii. **(10 points).** Prove Theorem 1 using (2b.i).

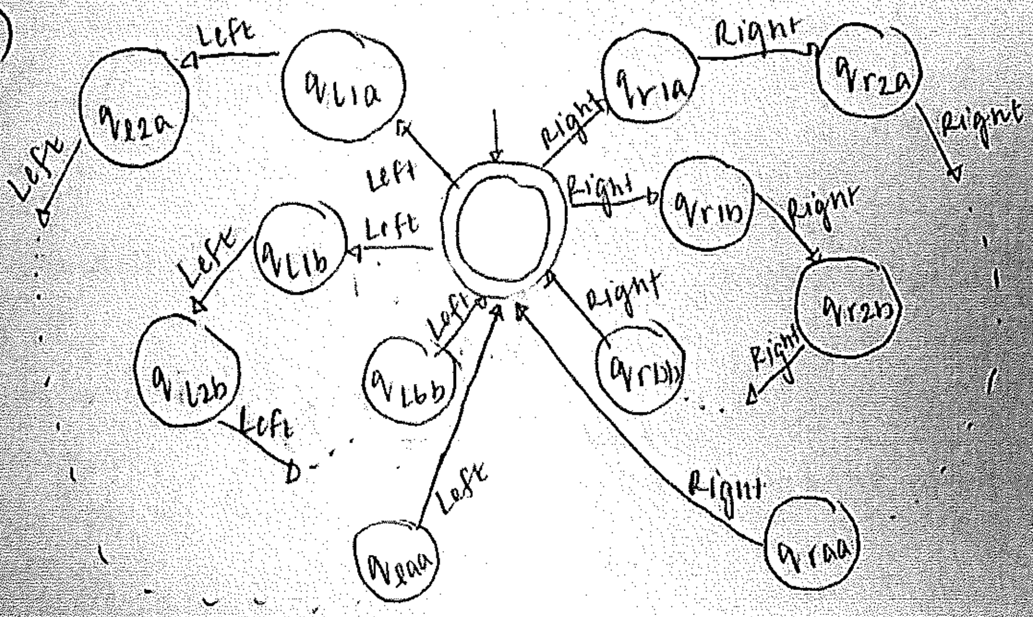
c) **(30 points).** Prove that for any NLR R, there exists an $n \in \mathbb{N}$ such that if R ever powers down at a position farther than n units from the origin, then there are an infinite set of positions at which R can power down. In other words, prove that for any NLR R, there exists an $n \in \mathbb{N}$ such that if there is a $k \in Loc(R)$ with $|k| > n$, then $Loc(R)$ is infinite. *Hint: Think of how we proved the pumping lemma.*

d) **Extra Credit (30 points).** An NLR that becomes Drone-Friended[1] operates the same as before except that when it reaches an accepting state, in addition to the option of doing nothing or powering down, it can instead choose to call its drone friend to immediately fly it back to the origin. Once back at the origin, it will continue operating as usual. Note that it will continue operating from the state that it was in when it called its drone friend, and will *not* reset its state to the starting state.

Let R be an NLR such that Loc(R) is of finite size n. If R becomes Drone-Friended, then what is the maximum size of the new location-set it now recognizes? Why? Give the answer as a function of n or list it as infinity. Then explain why.

a)

15



b) p. 9-10

[1]Not to be confused with "Friend-Droned", which is similar in that you think you're going somewhere, and just when you think you've been accepted, you get dumped back at square one.

Name: Eynn-Marie Phan
Student ID: 504 757 459

1c and d) Let m = # states. Let $n = m+1$

Suppose $|x| = \ell \geq n$ and $x \in L(F)$

Let the sequence of states of F on input x be $q_0, q_1, \dots q_\ell$

Let F read x in the order $x_1 x_2 \dots x_\ell$ so that on input character x_c, F transitions from q_{c-1} to q_c.

$\underbrace{q_0, q_1, \dots q_m, q_n, \dots}$ $\dots q_\ell$

by the pigeonhole principle, there is a repeated state

$\exists\, i, j$ s.t. $q_0, \dots q_{i-1} \dots q_j \dots q_n, \dots$ $i-1 < j \leq n$ and $q_{i-1} = q_j$

Therefore the sequence $q_{i-1}, \dots q_j$ is a cycle, and $L(F)$ is infinite because F accepts any string with a form such that F reads it in the order $x_1 \dots x_{i-1} (x_i \dots x_j)^k x_{j+1} \dots x_\ell$ $\forall k \geq 0$

The assignment of a, b, c, d, and e depends on
- the number of flipping states in the cycle. This is either even or odd.
- the orientation of the input string the 1st time F gets to q_i. This is either forward or backward.

(after this I got really confused about indexes/which things to reverse, so I'm not sure if the rest is right)

$x_1 x_2 \dots x_\ell$ is the order in which F reads y.
Unless there are no flipping states in $q_1 \dots q_\ell$, $x \neq x_1 x_2 \dots x_\ell$

Name: Erynn-Marie Phan
Student ID: 504 757 459

(1c) Case 1: the number of flipping states in the cycle $q_i \dots q_j$ is even

- Case 1a: the orientation of the input string the 1st time F gets to q_i is forward

Let a be composed of substrings $a_1, a_2, \dots a_k$ for some $k \geq 1$
Let e be composed of substrings $e_1, e_2 \dots e_{k-1}$ (if $k=1$, $e=\varepsilon$)
$x_1 x_2 \dots x_{i-1} = a_1 e_1 a_2 e_2 \dots e_{k-1} a_k$. Then $a = a_1 a_2 \dots a_k$ and $e^R = e_1 e_2 \dots e_{k-1}$

1a1: If there are 0 flipping states in $q_i \dots q_j$
Let $b = x_i \dots x_j$, let $d = \varepsilon$. Then $|b| + |d| > 0$ because $i < j$
Let $x_{j+1} \dots x_\ell = c_1 c_2 \dots c_k$ for some $k \geq 1$. Then $c = c_1 c_3 \dots c_k \, c_4^R c_2^R$ or $c_1 c_3 \dots c_k^R \dots c_4^R c_2^R$

Then $|x_1 \dots x_j| = |abde| \leq n$ since $j \leq n$

and $ab^\alpha c d^\alpha e \in L(F)$ $\forall \alpha \geq 0$

1a2: If there are p flipping states in $q_i \dots q_j$, where p is even and $p \geq 2$
$q_i \dots q_{f1} \; q_{f2} \dots q_{fp} \dots q_j$
flipping states

Let $b = x_i \dots x_{f1} x_{f2+1} \dots x_{f3} \dots x_{f(n-2)+1} \dots x_{f(n-1)} x_{fn+1} \dots x_j$
$d^R = x_{f1+1} \dots x_{f2} x_{f3+1} \dots x_{f4} \dots x_{f(n-1)+1} x_{fn}$

c = the same as case 1a1

(1a, case 1a continued)

Then $|b|+|d| > 0$ because there are 2 or more characters corresponding to flipping states.

$|abde| = |x_1 \dots x_{fn}| \leq n$ because $fn \leq j \leq n$

and $ab^\alpha c d^\alpha e \in L(F)$ $\forall \alpha \geq 0$ because F reads this input in the order $x_1 \dots x_{i-1} (x_i \dots x_j)^\alpha x_{j+1} \dots x_\ell$

- Case 1b: The number of flipping states is even, and the orientation of the input is backward

Let $a_1, \dots a_k$ and $e_1, \dots e_k$ be strings $\in \Sigma^*$

$x_1 \dots x_{i-1} = a_1 e_1 \dots a_k e_k$. $a = a_1 a_2 \dots a_k$ and $e^R = e_1 e_2 \dots e_k$

- If there are 0 flipping states in $q_i, \dots q_j$

30

$d^R = x_i \dots x_j$, $b = \varepsilon$

$x_{j+1} \dots x_\ell = c_1 c_2 \dots c_k$, $c = c_2 c_4 \dots c_k \dots c_3^R c_1^R$ or $c_2 c_4 \dots c_k^R \dots c_3^R c_1^R$

Then the conditions of lemma 1 hold for the same reasons as before

- If there are p flipping states in $q_i, \dots q_j$, p even, $p \geq 2$

$d =$ the same as b in case 1a2.

$b^R =$ the same as d^R in case 1a2.

Then the conditions of lemma 1 hold.

Case 2: the number of flipping states p in $q_i, \dots q_j$ is odd

- Case 2a: orientation @ q_i = forward

Define a and e in the same way as 1a.

$b = x_i \dots x_{f1} x_{f2+1} \dots x_{f3} \dots x_{f(n-1)+1} \dots x_{fn}$

$d^R = x_{f1+1} \dots x_{f2} x_{f3+1} \dots x_{f4} \dots x_{f(n-2)+1} \dots x_{f(n-1)} x_{fn+1} \dots x_j$

$c =$ the same as in case 1b

Name: Erynn-Marie Phan
Student ID: 504 757 459

Then the conditions of Lemma 1 hold.

- Case 2b: p is odd, orientation @ q_i = backward

Define a and c in the same way as 1b.

d = the same as b in 2a

b^R = the same as d^R in 2a

c_{π_i} = the same as in case 1a

Then the conditions of Lemma 1 hold for the same reasons as before.

d) 15

Name: Evynn-Marie Phan

Student ID: 504 757 459

2bi. $\exists$ a path through R that directs R to position k in m time steps $\Rightarrow$ $\exists$ a sequence of states $q_0, q_1, \dots q_m$ with L_n^{total} left transitions and R_n^{total} right transitions, where

$-L + R = k$

for every pair of repeated states (q_i, q_j) where $i<j$, there is a cycle $q_i, \dots q_j$

$q_0, q_1, \dots \underbrace{q_i, q_{i+1} \; q_j}_{\text{cycle}}, q_{j+1} \dots q_m \Rightarrow q_0, q_1, \dots q_i, q_{j+1} \dots q_m$ is also a valid path, w/ L' left transitions & R' right transitions

Let # left transitions in the cycle = l

" # right " " " " = r

20

R is unproductive $\Rightarrow$ $l + r = 0$ and

$-L' + R' = k$, so the shorter path still takes R to position k

We can do this for every pair of repeated states until there are none.

Therefore $\exists$ a path to position k with no repeated states.

Name: Erynn-Marie Phan
Student ID: 504 757 459

ii. by (i), for every $k \in Loc(R)$, $\exists$ a path to k w/ no repeated states.

Let # states = m. Then any $k \in Loc(R)$ can be reached using $\leq m$ states. $\Rightarrow$ using $\leq m$ transitions

Then any $k \in Loc(R)$ is at most m steps away from the origin 10

c.) intuition: If n is far enough from the origin (more than the number of states), there is a cycle in R that ~~is productive~~/ has an unequal # of Left and Right transitions. R can repeat the cycle any number of times and potentially get infinitely far away from the origin 20

correct intuition