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Quiz

CS170A – Mathematical Models & Methods for Computer Science

Spring 2013

OPEN BOOK

Wednesday, April 17, 2010, 5:00pm-5:50pm

Do not cheat.

Problem	*Points*
1	30 /32
2	30 /36
3	22 /32
Total	82 /100

1. **True/False**
For each of the following matrices, determine (yes/no) whether the matrix is: invertible, unitary, orthogonal, Hermitian. Recall A is Hermitian if $A' = A$, orthogonal if $A^{\mathsf{T}} = A^{-1}$, and unitary if $A' = A^{-1}$ (where $A' = \overline{A^{\mathsf{T}}}$ denotes the conjugate transpose), and $i = \sqrt{-1}$.

-2

matrix	invertible?	Hermitian?	orthogonal?	unitary?
$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$	True ☐ False ☑	True ☑ False ☐	True ☐ False ☑	True ☐ False ☑
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	True ☑ False ☐	True ☐ False ☑	True ☑ False ☐	True ☑ False ☐
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$	True ☑ False ☐	True ☐ False ☑	True ☐ False ☑	True ☐ False ☑ ✗
$\frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ i & -i \end{pmatrix}$	True ☑ False ☐	True ☐ False ☑	True ☐ False ☑	True ☑ False ☐

2. **True/False AND Example/Counterexample**
For each of the following equations, mark whether the equation is True (always valid) or False. Assume for real values θ that 2×2 rotation matrices and 2×2 reflection matrices have the forms $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ and assume that if $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the eigenvalues of X are $\lambda = \left((a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right)/2$.

If you mark an answer True, you must also give an example for full credit; if you mark an answer False, you must also give a counterexample.

(a) **True** ☑ **False** ☐ For each of the properties in the table above (invertible, Hermitian, orthogonal, unitary): if a matrix X has that property, then so does X'.

(counter)example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) **True** ☑ **False** ☐ If X is a 2×2 reflection matrix, then $X = X^{-1} = X'$.

(counter)example:
$X =$ Reflection $= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ $\quad X^{-1} = -1\begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = X$

$X' = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = X$

(c) **True** ☑ **False** ☐ If X is a real symmetric matrix with eigendecomposition $X = Q\,L\,Q'$ (where L is a diagonal matrix of eigenvalues and Q is a real orthogonal matrix), then $X^k = Q\,L^k\,Q'$ for any integer $k \geq 0$.

(counter)example:
$X = QLQ'$
$X^2 = (QLQ')(QLQ') = QL^2Q'$
$X^3 = X^2 \cdot QLQ' = QL^2Q'QLQ = QL^3Q'$ ...

$Q'Q = I$ b/c Q is orthogonal

-6 (d) **True** ☑ ✗ **False** ☐ If X is a real symmetric matrix, and X is unitary, then X is the identity matrix.

(counter)example:
$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{-2} & \frac{-1}{2} \end{bmatrix}$

(e) **True** ☑ **False** ☐ If X is a real symmetric matrix and is invertible, then the eigenvalues of X^{-1} are the inverses of eigenvalues of X.

(counter)example: $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda+1 \end{vmatrix}$ $\quad (\lambda-1)(\lambda+1) - 1 = 0$
$\lambda^2 - 2 = 0$
$\lambda = \frac{\pm\sqrt{4\cdot 2}}{2} = \pm\sqrt{2}$

$X^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) - \frac{1}{4} = 0$
$\lambda^2 - \frac{1}{2} = 0$
$\frac{\pm\sqrt{4\cdot\frac{1}{2}}}{2} = \pm\frac{\sqrt{2}}{2}$

(f) **True** ☐ **False** ☑ The determinant of a real orthogonal matrix Q is always nonnegative.

(counter)example: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

MATLAB

(a) You are given the following MATLAB functions:

```
function V = v1(x)
 n = length(x);  %  assume x is a vector of length n
 V = ones(n,n);
 for i=1:n
    for j=1:n
        V(i,j) = x(j)^(n-i);
    end
 end
```

```
function V = v2(x)
 n = length(x);  %  assume x is a vector of length n
 x = x(:);       %  make sure x is a n*1 column vector
 V = (x * ones(1,n)) .^ (ones(n,1) * (n-(1:n)));
```

i. What matrix does the invocation v1(1:3) produce?

$$\begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

ii. What matrix does the invocation v2(1:3) produce?
(Remember: when X and Y are matrices of the same size, then X .^ Y yields $Z = (z_{ij})$ where $z_{ij} = x_{ij}^{y_{ij}}$.)

$[1\ 4\ 9]$ × −4

iii. Is there a 3 × 1 vector x for which v1(x)' - v2(x) is nonzero?

Yes   example? −2.

(b) You are given the following MATLAB functions:

```
function x = spectral_norm(A)
 x = sqrt(eig(A' * A));  % x = square root( largest eigenvalue of A' A )
 x= sqrt(max(eig(A'*A)));
function x = max_norm(A)
 x = max(sum(abs(A)));   % x = largest row sum of abs(A)    (= |A|).

function demo(A)
 disp(sprintf("max_norm = %f    spectral_norm = %f\n", max_norm(A), spectral_norm(A)))
```

First, repair the definition of spectral_norm above (the comment is correct, but its code is incorrect), and complete the definition of max_norm above (replace the underscores with an expression).
With these changes, what would be the result of the MATLAB invocation demo(diag(1:3))?

max_norm = 3   spectral_norm = ~~9~~   −4

(c) You are given the following MATLAB function:

```
function t = spd(A)
 t = min( norm(A-A')==0, min(eig(A)>0) )
```

What result does the invocation spd(diag(1:3)) produce?
(Hint: the function min operates on boolean values in the way you'd expect.)

true