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**CS170A – Mathematical Models and Methods for Computer Science**
**Spring 2007**
**D.S. Parker**

**Midterm Examination**
**OPEN BOOK, OPEN NOTES**
**Thursday, May 10, 2007, 4:00pm-5:50pm**

Do not cheat.

| *Problem* | *Points* |
|---|---|
| 1 | 19 /25 |
| 2 | 25 /25 |
| 3 | 25 /25 |
| 4 | 25 /25 |
| *Total* | 94/100 |

1. **Everything**

For each of the following matrices, determine (yes/no) whether the matrix is: unitary, hermitian, invertible, positive definite. (Recall $A$ is positive definite if it is Hermitian and all of its eigenvalues are positive real values.) Assume that $i = \sqrt{-1}$.

| | matrix | unitary? | hermitian? | invertible? | positive definite? |
|---|---|---|---|---|---|
| det $0-2=-2$ | $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ | **True** ☐ **False** ☒ | **True** ☐ **False** ☒ | **True** ☒ **False** ☐ | **True** ☐ **False** ☒ |
| det $= -1$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | **True** ☐ **False** ☒ | **True** ☒ **False** ☐ | **True** ☒ **False** ☐ | **True** ☐ **False** ☒ |
| det $-4$ | $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ | **True** ☐ **False** ☒ | **True** ☒ **False** ☐ | **True** ☒ **False** ☐ | **True** ☐ **False** ☒ |
| det $= 3-2$ $=1$ | $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ | **True** ☒ **False** ☐ | **True** ☐ **False** ☒ | **True** ☒ **False** ☐ | **True** ☐ **False** ☒ |

For each of the following equations, mark whether the equation is True (valid for all specified matrices $A$) or False. Assume that $A'$ denotes the transpose of $A$, $A'$ denotes the hermitian transpose of $A$, and $i = \sqrt{-1}$.

For full credit, if you mark it True, you must explain how you derived this. If you mark it False, you must give a *counterexample* (values for these sets for which the equation does not hold).

(a) **True** ☒ **False** ☐

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the eigenvalues of $A$ are $\lambda = \left((a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right)/2$

**Explanation:** This is the soln to the eq'n $\det(\lambda I - A) = 0$

see previous page

(b) **True** ☐ **False** ☒

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\operatorname{tr}(A) = a + d$ (= the sum of the diagonal elements of $A$) is the same as the sum of the eigenvalues of $A$.

**Explanation:** This is true if $A$ has real eigenvalues. But in $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ they are not.

$-4$

(c) **True** ☐ **False** ☒

Given a rotation matrix $Q = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ with rotation angle $\theta$, then $Q^n$ is a rotation matrix with rotation angle $n\,\theta$.

$\sin^2\theta = 1 - \cos^2\theta$

**Explanation:** if $n = 2$, $Q^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta - \sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 2\cos^2\theta - 1 \end{pmatrix}$

(d) **True** ☒ **False** ☐

If $A = \operatorname{cov}(X)$ is a covariance matrix, then its eigenvalues are real and nonnegative.

**Explanation:** Since $\operatorname{cov}(X)$ is of the form $M'M$ for some $[M]_{n\times m}$, then if $M = USV'$, $M'M = VS'U'USV'$
$= VS^2V'$

2. **SVD**

Assume that $X$ is a real matrix, and that $A$ is the covariance matrix $\mathrm{cov}(X)$, so

$$A = \mathrm{cov}(X) = \frac{1}{n-1}(X-M)'(X-M)$$

where $M$ is the matrix whose columns are filled with means of corresponding columns in $X$.

(a) Show that $A$ must have an SVD of the form

let $X - M = B$.

$$A = Q\,D\,Q',$$

where $D$ is a diagonal matrix of nonnegative real values and $Q$ is orthogonal.

By SVD, $B = USV'$, where $S$ is diagonal & $U, V$ are orthogonal,
↳ & nonnegative real values.

then $B'B = VS'U'USV'$
$U'U = I$ since its orthogonal
$= VS^2V' \sim QDQ'$ with these properties

(b) Assuming again that $A = Q\,D\,Q'$, where $D$ and $Q$ are as in part (a), show that $A$ is positive definite.

$D$ contains the eigenvalues of $A$, and all are positive real values.
Since all matrices of the form $M'M$ are symmetric (assuming real values)
and $A \sim (X-M)'(X-M)$ with real values, $A = A^T$. $\Rightarrow$ $A$ is Hermitian.
$A$ is Hermitian and all eigenvalues are positive real $\Rightarrow$ $A$ is pos. definite.

(c) Assuming again that $A = Q\,D\,Q'$, and using the fact that $\det(XYZ) = \det(X)\det(Y)\det(Z)$ for square matrices $X$, $Y$, $Z$, give a formula for $\det(A)$ in terms of $\det(D)$.

$\det(A) = \det(Q)\det(D)\det(Q')$

since $Q$ and $Q'$ are orthonormal, $\det(Q) = \det(Q') = 1$

$\det(A) = 1 \cdot \det(D) \cdot 1$
$= \det(D)$

(d) Assuming that $A = \mathrm{cov}(X)$, is $A$ Hermitian? If yes, show why; if not, give a counterexample.

Yes, because it is the product of a matrix with its conj. transp, which is a symmetric matrix.
Since $A$ is symmetric, it must be Hermitian since $A = A'$.

(e) Assuming again that $A = Q\,D\,Q'$, what is the SVD of $A^{-1}$?

$A^{-1} = (QDQ')^{-1}$
$= (Q')^{-1}D^{-1}Q^{-1}$
$= Q\,D^{-1}Q'$ since $Q, Q'$ are orthogonal, and $Q' = Q^{-1}$

3. **Least Squares**

Gauss became famous because he was able to fit an ellipse through the observed positions of the asteroid Ceres using least squares. Your job here is to show how he obtained this fit.

Suppose you are given a table of $n$ pairs of real $(x, y)$ values

| $x$ | $y$ |
|---|---|
| $x_1$ | $y_1$ |
| $\vdots$ | $\vdots$ |
| $x_n$ | $y_n$ |

and you are asked to fit the line

$$a x^2 + b y^2 = 1$$

through this data, finding real coefficients $a$ and $b$ that minimize squared error.

(a) Express this as a linear problem. That is, give a matrix $A$ whose entries are functions of $\{x_i\}$ and $\{y_i\}$ for which $a$ and $b$ can be obtained by least squares on $A\begin{pmatrix} a \\ b \end{pmatrix} = \vec{1}$, where $\vec{1}$ is the $n \times 1$ vector whose entries are all 1.

$$A = \begin{pmatrix} x_1^2 & y_1^2 \\ \vdots & \vdots \\ x_n^2 & y_n^2 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = A \underset{\text{fancy}}{\backslash} \vec{1}$$

✓

(b) Give an explicit formula for $A'A$ (in terms of $\{x_i\}$ and $\{y_i\}$).

$$A'A = \begin{pmatrix} \sum x_i^4 & \sum x_i^2 y_i^2 \\ \sum x_i^2 y_i^2 & \sum y_i^4 \end{pmatrix}$$

✓

$(1+2)(3+4)$

(c) Give an explicit formula for $a$ (in terms of $\{x_i\}$ and $\{y_i\}$).

$$a = \underbrace{(A'A)^{-1} A' (\vec{1})}$$

$$A'(\vec{1}) = \begin{pmatrix} x_1^2 & \cdots & x_n^2 \\ y_1^2 & \cdots & y_n^2 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum x_i^2 \\ \sum y_i^2 \end{pmatrix}$$

$$(A'A)^{-1} A' = \frac{1}{\sum x_i^4 \sum y_i^4 - 2\sum x_i^2 y_i^2} \begin{pmatrix} \sum y_i^4 & -\sum x_i^2 y_i^2 \\ -\sum x_i^2 y_i^2 & \sum x_i^4 \end{pmatrix}$$

$$a = \frac{\sum y_i^4 \sum x_i^2 - \sum x_i^2 y_i^2 \sum y_i^2}{\sum x_i^4 \sum y_i^4 - 2\sum x_i^2 \sum y_i^2}$$

ok

4. **PCA**

Consider the following list of polyhedra and their properties:

| Polyhedron | Faces | Vertices | Edges |
|---|---|---|---|
| tetrahedron | 4 | 4 | 6 |
| octahedron | 8 | 6 | 12 |
| cube | 6 | 8 | 12 |
| dodecahedron | 12 | 20 | 30 |
| icosahedron | 20 | 12 | 30 |

With Matlab we produce following session:

```
>> X = [
          4    4     6 ;
          8    6    12 ;
          6    8    12 ;
         12   20    30 ;
         20   12    30 ;
       ];

>> cov(X)
ans =
    40    23    63
    23    40    63
    63    63   126

>> [U,S,V] = svd(cov(X))
U =
   -0.4082    0.7071   -0.5774
   -0.4082   -0.7071   -0.5774
   -0.8165   -0.0000    0.5774
S =
  189.0000         0         0
         0   17.0000         0
         0         0    0.0000
V =
   -0.4082    0.7071   -0.5774
   -0.4082   -0.7071   -0.5774
   -0.8165   -0.0000    0.5774
>>
```

(a) Identify the first two principal components of the dataset $X$.

$U(:,1)$ and $U(:,2) = (.7071;\ -.7071;\ 0)$

$\hookrightarrow = (-.4082;\ -.4082;\ -.8165)$

(b) Explain how to project the dataset $X$ onto these first two components.

The dot product of each with X. ex. $\begin{bmatrix} 4 & 4 & 6 \\ 8 & 6 & 12 \\ 6 & 8 & 12 \\ 12 & 20 & 30 \\ 30 & 12 & 30 \end{bmatrix} \begin{bmatrix} -.4082 \\ -.4082 \\ -.8165 \end{bmatrix} = [\_\ \_\ \_\ \_\ \_]$

(c) Suppose we want to find a linear relationship between the number of Faces, Vertices, and Edges. That is, we want to find real values $a$, $b$, $c$ such that

$$\mathit{Faces} \quad = \quad a \cdot \mathit{Vertices} \quad + \quad b \cdot \mathit{Edges} \quad + \quad c$$

Explain how these equations could be solved using least squares, by converting them to matrix form and explaining what matrix operations you could use to find $a$, $b$, and $c$.

Use $A = \begin{pmatrix} V_1 & E_1 & 1 \\ \vdots & \vdots & \vdots \\ V_n & E_n & 1 \end{pmatrix}$ then $A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (A'A)^{-1} A' \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$

(d) The last singular value is zero; what does this mean about ~~the value of c~~? the dataset X

That means the third (edges) follows from the first two, faces & vertices.

This implies that you only need
the first 2 PC's to reconstruct the dataset.