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UCLA - Henry Samueli School of Engineering
Department of Computer Science
CS112 - Midterm
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Do not turn until you have been told to.

Exercise	Points	Grade
1	20	20
2	10	10
3	20	~~10~~ 20
4	20	20
5	20	20
6	10	10

~~80~~

100

F.A

1 20 Points

Suppose you do not remember your Gmail password. You know that it contains either 6 or 7 characters, and that each character it contains is either a lowercase letter {a,^{26}...,z} or a numerical digit {0, . ^{10}. . , 9}. As far as you know, all strings that satisfy these conditions are equally likely to be your password. For this problem, you may leave your answers in terms of arithmetic operations, such as addition, multiplication, and factorials.

- Find the following values first:
  - S_6 the cardinality of the set of all possible passwords of length 6
  - S_7 the cardinality of the set of all possible passwords of length 7
  - p_6 the probability that the password is of length 6
  - p_7 the probability that the password is of length 7
  - p_a the probability that the password is alphabetical
  - p_n the probability that the password is numerical
- You sit down at the computer and start guessing strings of length 6 or 7 containing only lowercase letters and digits uniformly at random and independently, without keeping track of the strings you have already guessed. What is the probability that you guess the correct password in your first 1000 tries?
- What is the probability that your password contains exactly two numerical digits?
- What is the probability that all of the characters in the password are unique (i.e., no letter or digit appears in the password more than once)?

$$S_6 = (26+10)^6 = 36^6$$

$$S_7 = (26+10)^7 = 36^7$$

$$p_6 = \frac{S_6}{S_6+S_7} = \frac{36^6}{36^6+36^7}$$

$$p_7 = \frac{S_7}{S_6+S_7} = \frac{36^7}{36^6+36^7}$$

$$p_a = \frac{26^6+26^7}{36^6+36^7}$$

✓

$$p_n = \frac{10^6+10^7}{36^6+36^7}$$

$P[\text{randomly guess correctly in 1000 indep. trials}] = 1 - P[\text{Never guess correctly in 1000 trials}]$

$= 1 - P[\text{Guess wrong 1000 times}]$

$= 1 - \left(1 - P[\text{guess the one}]\right)^{1000}$

$= 1 - \left(1 - \frac{1}{36^6 + 36^7}\right)^{1000}$

$P[\text{password contains 2 numerical}] = P[\text{2 numerical \& length 6}] + P[\text{2 numerical \& Length 7}]$

$= \dfrac{10^2 \cdot 26^4 \cdot \binom{5}{2} + 10^2 \cdot 26^5 \binom{6}{2}}{36^6 + 36^7}$

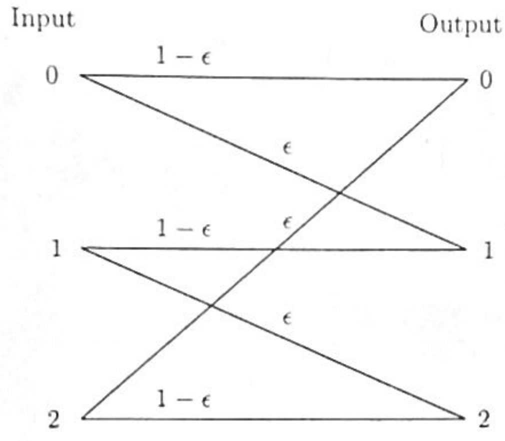
_ A _ A _ A _ A _

$= \dfrac{10^2 \cdot 26^4 \cdot \frac{5!}{3! \cdot 2!} + 10^2 \cdot 26^5 \, \frac{6!}{4! \cdot 2!}}{36^6 + 36^7}$

$P[\text{all chars unique}] = \dfrac{\frac{36!}{(36-6)!} + \frac{36!}{(36-7)!}}{36^6 + 36^7}$

2 10 Points

A ternary communication channel is shown in the following figure.



Ternary Communication Channel

The input symbols 0, 1, and 2 occur with probabilities 1/2, 1/4, and 1/4, respectively.

- Find the probability P_0=P[the output is 0], in terms of ϵ
- Suppose the observed output is 0. Find the probability that the input was 0.
- Find the probability of error, i.e., the probability that the output symbol is different from the input symbol, in terms of ϵ.

$$P_0 = P[\text{Out } 0 \mid \text{In } 0]\, P[\text{In } 0] + P[\text{Out } 0 \mid \text{In } 1]\, P[\text{In } 1] + P[\text{Out } 0 \mid \text{In } 2]\, P[\text{In } 2]$$

$$= (1-\epsilon)(\tfrac{1}{2}) + (0)(\tfrac{1}{4}) + (\epsilon)(\tfrac{1}{4})$$

$$\boxed{= \tfrac{1}{2} - \tfrac{1}{4}\epsilon}$$

$$P[\text{In } 0 \mid \text{Out } 0] = \frac{P[\text{In } 0 \ \& \ \text{Out } 0]}{P[\text{Out } 0]} = \frac{P[\text{Out } 0 \mid \text{In } 0]\, P[\text{In } 0]}{P[\text{Out } 0]}$$

$$= \frac{(1-\epsilon)(\frac{1}{2})}{\frac{1}{2} - \frac{1}{4}\epsilon}$$

$$\boxed{= \frac{2-2\epsilon}{2-\epsilon}}$$

$$P[\text{Input} \neq \text{Output}] = P[I \neq 0 \mid \text{In } 0]\, P[\text{In } 0] + P[I \neq 0 \mid \text{In } 1]\, P[\text{In } 1] + P[I \neq 0 \mid \text{In } 2]\, P[\text{In } 2]$$

$$= \tfrac{1}{2}\epsilon + \tfrac{1}{4}\epsilon + \tfrac{1}{4}\epsilon$$

$$\boxed{= \epsilon}$$

3 20 Points

Consider n independent streams of packets arriving at a router to be merged on an output link. Let X_i be the number of packets that arrive on link i in a time interval of 1 second, and let X_i be Poisson(α_i). Let the total number of messages that depart during one second, $Y = X_1 + X_2 + ...X_n$.

- Derive the pmf of Y. $G(z) = e^{-\lambda(1-z)}$
- What is the expected number of packets that depart in 10 seconds.

$Y = X_1 + X_2 \cdots + X_n$

$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} e$

Z transform

$G_Y(z) = G_{X_1}(z)\, G_{X_2}(z) \cdots G_{X_n}(z)$

$z=0$

$= e^{-\alpha_1} e^{-\alpha_2} \cdots e^{-\alpha_n}$

$= e^{-\sum_{i=1}^{n} \alpha_i}$

pmf $Y = \text{Poisson}\left(\sum_{i=1}^{n} \alpha_i\right)$

? 10

F.A.

Poisson:
$E[X] = \alpha$

10

$E[\text{Packets in 10 seconds}] = \sum_{i=1}^{n} 10 \cdot \alpha_i$

$= \boxed{10 \cdot \sum_{i=1}^{n} \alpha_i}$

4 20 Points

A computer system consists of n subsystems, each of which has an exponential distribution of lifetime with parameter λ_i , $i = 1, 2, , n$. Each subsystem is independent; the whole computer system fails if any one of the subsystems fails.

- Find the distribution of the minimum of these lifetimes.
- Suppose that $\lambda_1 = \lambda_2 = ... = \lambda_n$. How does the probability change when n increase? In other words how does the probability change if we keep adding subsystems?
- What is the probability that subsystem i fails?

$$F(t) = P[Y \geq y] = 1 - P[X_1 \geq x] + P[X_2 \geq x] + \dots + P[X_n \geq x]$$

$$= e^{-\lambda_1 t} e^{-\lambda_2 t} \dots e^{-\lambda_n t}$$

$$= 1 \, e^{-t(\sum_{i=1}^{n} \lambda_i)}$$

$$\text{dist} = \exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

The probability decreases as n increases $\rightarrow \frac{1}{e^{\lambda t}}$ 5

$$\int_0^\infty \lambda_i e^{-t(\sum_{i=1}^{n} \lambda_i)} \, dt$$

$$= \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} \left(e^{-t \sum_{i=1}^{n} \lambda_i} \right) \Big|_0^\infty$$

$$= \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} (0 - (-1))$$

$$= \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

$$= \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$$

5 20 Points

A datagram subnet allows routers to drop packets whenever they need to. The probability of a router discarding a packet is p. Consider the following network:

$$Source \overset{p}{-} -SourceRouter \overset{p}{-} -DestinationRouter \overset{p}{-} -Destination$$

If both host-router and router-router lines are counted as hops,

- Let $p(k)$ be the probability that a packet is dropped at the k^{th} hop. Express $p(k)$ in terms of p, for k=1,2,3.
- Determine the expectation $E[k]$.

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$p(1) = P[\text{1st hop drops}] = p$

$p(2) = P[\text{1st hop does not drop \& 2nd hop drops}] = (1-p)(p)$

$p(3) = P[\text{1st not drop \& 2nd not drop \& 3rd drop}] = (1-p)^2(p)$

$\boxed{p(k) = (1-p)^{k-1}p \quad k=1,2,3}$

$E[K] = \frac{d P(z)}{dz}\Big|_{z=1} = \frac{d}{dz}\frac{pz}{1-qz}\Big|_{z=1}$

$= \frac{p(1-qz) - (-q)(pz)}{(1-qz)^2}\Big|_{z=1}$

$= \frac{p(1-q)+qp}{(1-q)^2}$ $\quad q = 1-p$

$= \frac{p^2+p-p^2}{p^2}$

$\boxed{= \frac{1}{p} \quad \text{for } k = [1, \infty)}$

For $k = 1, 2, 3$

$(p-p^2)(1-p)$

$= p - p^2 - p^2 + p^3$

$E[K] = 1p + 2(p-p^2) + 3(p-2p^2+p^3)$

$\boxed{= 6p - 8p^2 + 3p^3 \quad \text{for } k=1,2,3}$

6 10 Points

The daily attendance to a class is of the form $C = 2M+3M^2$. M is distributed as a Poisson, find the expected value of C.

M has parameter λ

$E[C] = 2E[M] + 3E[M^2]$

$E[M] = \lambda$

$Var[M] = E[M^2] - E[M]^2$

$E[M^2] = Var[M] + E[M]^2$

$= \lambda + \lambda^2$

$E[C] = 2\lambda + 3(\lambda + \lambda^2)$

$= 5\lambda + 3\lambda^2$