1. **Bayes rule [6pt]**

   A packet arrives at a router and will be sent to 2 outgoing links. Link 1 is lossy so the probability of packet-loss is $\frac{2}{3}$. Link 2 is loss free. Suppose the packet will go to link 1 with probability $\frac{1}{4}$ and link 2 with $\frac{3}{4}$. What is the conditional probability that the packet goes to link 2 given that it is not lost?

   Answer

$$P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|K^C)P(K^C)}$$

$$P(K|C) = \frac{9}{10}$$

2. **TCP Window Size [6pt]**

In a certain version of TCP congestion control schemes, the congestion window size X is doubled every time a packet is delivered successfully. Let n denote the number of packets delivered successfully before packet loss occurs. Initially the window size X is set to be 1 (before the first packet is sent). Assume that the probability of packet loss each time is 0.5.

(a) [3pt] Find the pmf of n.
Solution:
The number of packets successfully delivered before loss occurs follows the geometry distribution.

$$p(n) = 0.5^n \cdot 0.5 = 0.5^{n+1}$$

where $n = 0, 1, 2...$

(b) [3pt] What is the expected window size E[X]?
Solution:

$$E[X] = \sum_{n=0}^{\infty} p(n) \cdot 2^n = \sum_{n=0}^{\infty} 0.5^{n+1} \cdot 2^n = \sum_{n=0}^{\infty} 0.5 = \infty$$

3. **Probability Inequalities [8pt]**

Label the following statements with either =, $\leq$ or $\geq$. The statements can be labeled with = if equality always holds, $\leq$ or $\geq$ if one of these hold. If no such inequality or equality holds, label the statement as NONE.

(a) [2pt] $P(B)$ versus $1 - [P(B^c, A) + P(B^c, A^c)]$.
Answer:$P(B) = 1 - [P(B^c, A) + P(B^c, A^c)]$

(b) [2pt] $p_Y(g(x))$ versus $p_X(x)$ if $Y = g(X)$, where X, Y are discrete random variable, $g(\cdot)$ is a function.
Answer: $p_Y(g(x)) \geq p_X(x)$ since $p_Y(g(x)) = \sum_{u:g(u)=g(x)} p_X(u)$

(c) [2pt] Probability mass function $p_X(x)$ versus $p_{XY}(x, y)$ for discrete random variables.
Answer:$p_{XY}(x, y) = p_X(x)p_{Y|X}(y|x) \leq p_X(x)$

(d) [2pt] Probability density function $f_X(x)$ versus $f_{XY}(x, y)$ for continuous random variables.
Answer: $f_{XY}(x, y) = f_X(x)f_{Y|X}(y|x)$. Since $f_{Y|X}(y|x)$ can be any value, so NONE.

4. **Splitting Poisson Distribution [10pt]**

The number of tasks arrive at a router is a Poisson (λ) variable. The router independently routes to computers A or B with probability p and $1-p$ respectively. Let X_N be the total number of tasks, X_A the number of tasks at computer A, and X_B the number of tasks at computer B.

(a) [4pt] Derive the pmf of X_A, X_B.

(b) [1pt] State the condition for two random variables to be independent.

(c) [3pt] Show that X_A and X_b are independent based on you answer on (a) and (b).

(d) [2pt] Find the covariance of X_N and X_A.

Solution: (a) **Page 73 of Course Reader.**

$$P(A=a) = \frac{e^{-\lambda p}\lambda p^a}{(a)!}, P(B=b) = \frac{e^{-\lambda(1-p)}\lambda(1-p)^b}{(b)!}$$

(b)

$$P_{A,B}(a,b) = P_A(a)P_B(b)$$

(c)

$$P(A=a,B=b) = \tfrac{e^{-\lambda}\lambda^{(a+b)}}{(a+b)!}\begin{pmatrix} a+b \\ a \end{pmatrix} p^a(1-p)^b$$
$$= P(A=a)P(B=b) = \tfrac{e^{-\lambda p}\lambda p^a}{(a)!}\tfrac{e^{-\lambda(1-p)}\lambda(1-p)^b}{(b)!}$$

(d)

$$\mathrm{cov}(N,A) = \mathrm{cov}(A+B,A) = \mathrm{cov}(A,A) + \mathrm{cov}(B,A)$$
$$= \mathrm{var}(A) = \lambda p$$

5. **Traffic of Wireless Communications [8pt]**

We model the number of wireless connections to a base station as Geometric distribution with parameter p. The traffic per connection is Poisson(λ).

(a) [4pt] Find the Z-transform of total traffic at the base station.

(b) [4pt] Find the first and second moment of total traffic using the Z-transform you found. Calculate the numerical values when $p = 0.1$ and $\lambda = 100$.

Solution:
The Z-transform of total traffic is

$$G_Y(z) = G_N(G_X(z)) = \frac{pz}{1-(1-p)z}\Big|_{z=e^{-\lambda(1-z)}} = \frac{pe^{-\lambda(1-z)}}{1-(1-p)e^{-\lambda(1-z)}} = \frac{p}{e^{\lambda(1-z)}-(1-p)}$$

The first and second moment are

$$\frac{dG_Y(z)}{dz}\Big|_{z=1} = \frac{\lambda p e^{-\lambda(1-z)}}{\left[e^{\lambda(1-z)}-(1-p)\right]^2}\Big|_{z=1} = \frac{\lambda}{p} = 1000$$

$$\frac{d^2G_Y(z)}{dz^2}\Big|_{z=1} = \frac{-\lambda^2 p e^{-\lambda(1-z)}\left[e^{\lambda(1-z)}-(1-p)\right]^2 + \lambda p e^{-\lambda(1-z)} 2\lambda\left[e^{\lambda(1-z)}-(1-p)\right]e^{\lambda(1-z)}}{\left[e^{\lambda(1-z)}-(1-p)\right]^4}\Big|_{z=1}$$

$$= \frac{2\lambda^2}{p^2} - \frac{\lambda^2}{p}$$

$$E[X^2] = \frac{\lambda}{p} + \frac{2\lambda^2}{p^2} - \frac{\lambda^2}{p} = 1901000$$