CS112 Midterm Exam
Spring 2015

Name: Mario Tinoco

UID: 404 500 049

Problem	Score
1	8
2	6
3	6
4	1
5	2
6	6
Total	29

- The exam is closed book, closed notes.
- Calculator allowed. No other electronic devices allowed.
- One cheat sheet allowed

1. (8 points) Assume there are three major computer vision algorithms that are designed to detect objects. 45% of researchers prefer using algorithm A, 40% researchers prefer using algorithm B, 15% prefer using algorithm C. For a certain test, algorithm A has a precision of 70%, algorithm B has a precision of 90%, and algorithm C has a precision of 55%. Precision here refers to the probability that an algorithm leads to correct results.

(a) (5 points) What is the probability that a random researcher has a correct result in this test using his preferred algorithm?

(b) (3 points) A researcher conducts the test using his preferred algorithm, and yields correct result. What is the probability that this researcher is using algorithm A?

	P	A
$P(A) = .45$		.70
$P(B) = .40$		.90
$P(C) = .15$		.55

(a) $= .45(.70) + .4(.9) + .15(.55)$

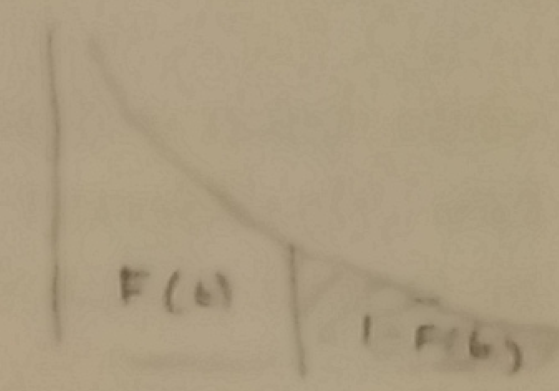
$= \boxed{75.75\%}$

(b) $P(A \mid \text{correct}) = \dfrac{P[\text{correct} \mid A]\, P[A]}{P[\checkmark \mid A] P[A] + P[\checkmark \mid B] P[B] + P[\checkmark \mid C] P[C]}$

$= \dfrac{(0.7)(0.45)}{.7575}$

$= \boxed{41.6\%}$

2. (8 points)



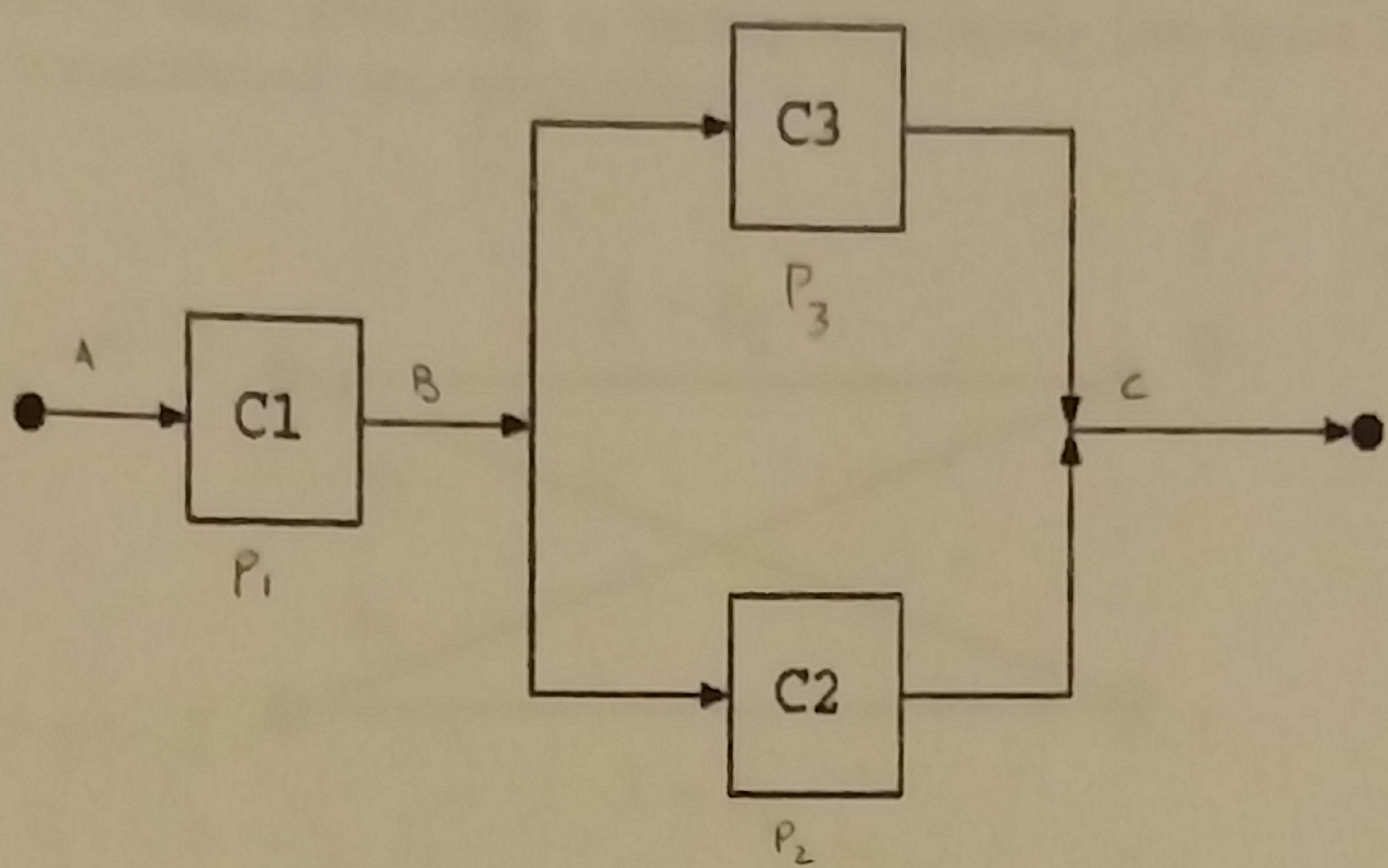


FIGURE 1. System configuration

(a) (4 points) What is the reliability for the system configuration shown in Fig.1? Assume the reliabilities for components C1, C2, and C3 are respectively p_1, p_2, and p_3. (Note that the interpretation of reliability here is the probability that, at a random time, the component is operational. The reliabilities of components are all mutually independent. There is no "lifetime" of the system in this view)

(b) (4 points) Now assume that we have lifetime distributions $F_1(t), F_2(t), F_3(t)$ respectively for C1, C2 and C3, i.e. $F_1(t) = P(T_1 < t)$ denotes the probability that the lifetime of C1 (T_1) is shorter than t. Give an expression in terms of $F_1(t), F_2(t), F_3(t)$ for the distribution of the time until the system fails $F(t)$ (i.e. the lifetime of the whole system).

4 (a) $R_{B\to C} = 1-(1-p_2)(1-p_3)$, $R_{A\to B} = p_1$, $R_{A\to C} = R_{A\to B}\cdot R_{B\to C} = p_1\left[1-(1-p_2)(1-p_3)\right]$

$$\boxed{= p_1 p_2 + p_1 p_3 - p_1 p_2 p_3}$$

(b)

2 $$F(t) = (1-F_1(t))(1-F_3(t)) + (1-F_1(t))(1-F_2(t)) - (1-F_1(t))(1-F_2(t))(1-F_3(t))$$

$$\boxed{= [1-F_1(t)][1-F_2(t)F_3(t)]}$$

you solved for $P(T>t)$ instead of $P(T<t)$

6

3. (8 points) A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$ respectively, and is received incorrectly with probability ϵ_0 and ϵ_1 respectively (see figure below). Errors in different symbol transmissions are independent.

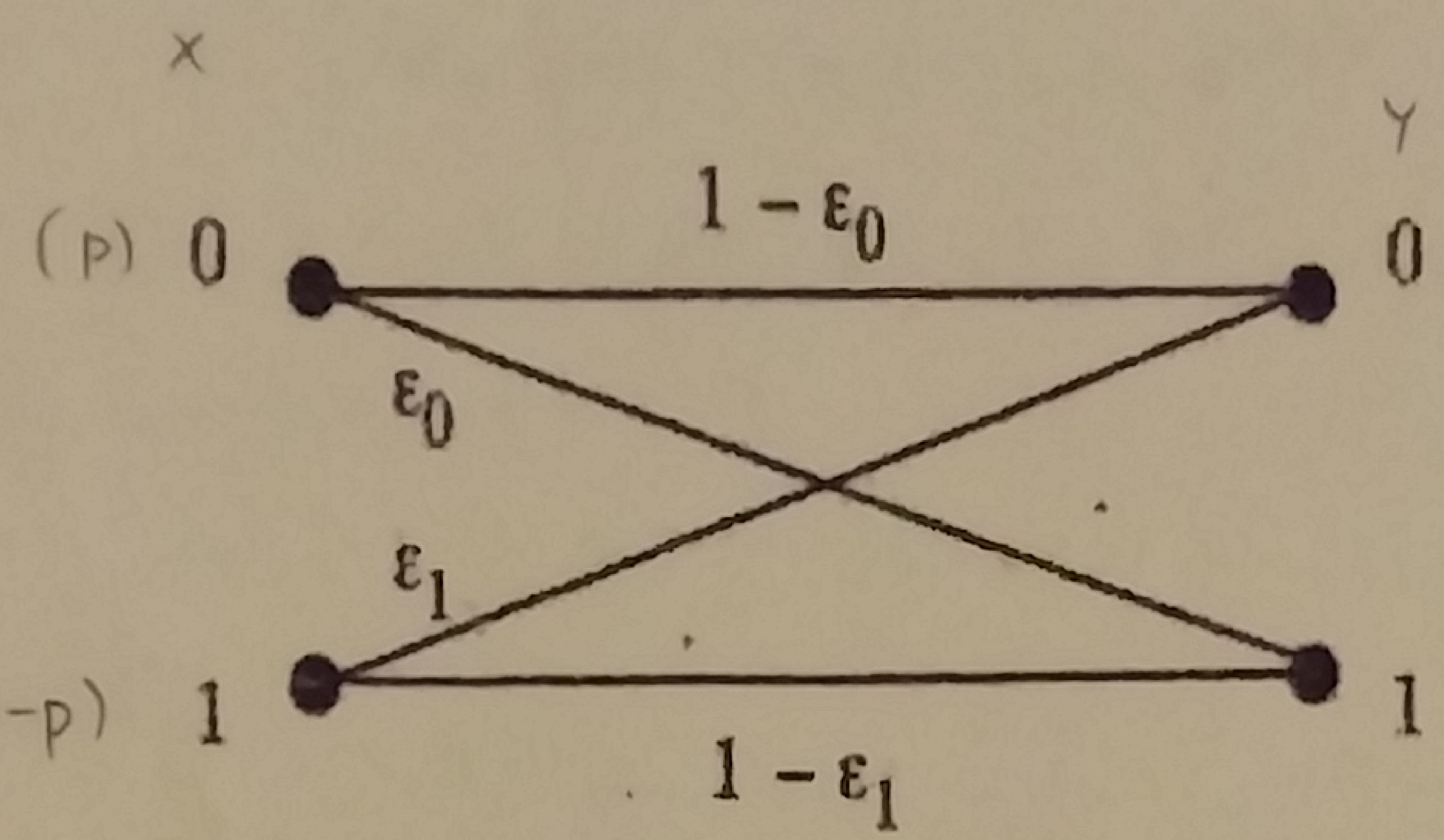


FIGURE 2. Error probabilities in a binary communication channel

(a) (3 points) What is the probability that a certain symbol is received correctly?

(b) (2 points) A string of symbols "1011" is sent. What is the probability that it is received correctly?

(c) (2 points) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. For example, a 0 is transmitted as 000 (or a 1 is transmitted as 111), and is decoded as 0 if and only if the received three-symbol string contains at least two 0s. What is the probability that 0 is received correctly?

(d) (1 points) For what values of ϵ_0 is there an improvement in the probability of correct decking of a 0 when the scheme in part (c) is used?

1

(a) P(0 received correctly) = $\boxed{1-\varepsilon_0}$ P(1 received correctly) = $\boxed{1-\varepsilon_1}$

P(symbol received correctly) = $p(1-\varepsilon_0) + (1-p)(1-\varepsilon_1)$

(b) P(1011 r.c.) = P(1) P(0) P(1) P(1) = $\boxed{(1-\varepsilon_1)^3(1-\varepsilon_0)}$

2

(c) P(# of 0's is ≥2) = $\binom{3}{2}(1-\varepsilon_0)^2(\varepsilon_0) + \binom{3}{3}(1-\varepsilon_0)^3(\varepsilon_0)^0$

2

$= \boxed{3\varepsilon_0(1-\varepsilon_0)^2 + (1-\varepsilon_0)^3}$

(d) $3\varepsilon_0(1-\varepsilon_0)^2 + (1-\varepsilon_0)^3 > 1-\varepsilon_0$

1

$3x^3-6x^2+3x - x^3+3x^2-3x+1 > 1-x$

$2x^3-3x^2+1 - 1+x > 0$

$2x^3-3x^2+x > 0$

$x(2x^2-3x+1) > 0$

$x(x-1)(2x-1) > 0$

6

Better when $0 < \varepsilon_0 < \frac{1}{2}$

4. (5 points) A computer system consists of n subsystems, each of which has an exponential distribution of lifetime with parameter $\lambda_i, i = 1, 2, ..., n$. Each subsystem is independent but the whole computer system fails if any of the subsystems do. Find the distribution of the minimum of these lifetimes.

X_i = lifetime of i sys. $\rightarrow F_{X_i}(x) = P[X_i \leq x] = 1 - e^{-\lambda_i x}$ +1

$Y = \min(X_1, X_2, X_3, \ldots) \rightarrow F_Y(x) = P[Y \leq x] = P[\min(x_1, x_2, \ldots) \leq x]$

$= P[X_1 \leq x \cup X_2 \leq x \cup X_3 \leq x \cup \ldots]$

$= \bigcup_{i=1}^{n} P[x_i \leq x]$

$$F_Y(x) = \boxed{\bigcup_{i=1}^{n} [1 - e^{-\lambda_i x}]}$$

5. (5 points) A *series* system is composed of two components with lifetime distribution $\mathrm{Exp}(\lambda_1)$ and $\mathrm{Exp}(\lambda_2)$ respectively. What is the probability that component 2 causes the system failure, assuming that the component lifetimes are independent.

— $\boxed{\mathrm{Exp}(\lambda_1)}$ — $\boxed{\mathrm{Exp}(\lambda_2)}$ —

X_1 = lifetime of 1 $\sim \mathrm{Exp}(\lambda_1)$ $\qquad$ $P[\text{2 causes failure}]$

X_2 = lifetime of 2 $\sim \mathrm{Exp}(\lambda_2)$ $\qquad$ $= P[X_2 < X_1]$ $+1$

$$= \lim_{h \to 0} P[X_2 < y \mid y \le X_1 \le y+h] \quad +1$$

$$P[X_2 < X_1] = \int \frac{F_{X_2}(y) f_{X_1}(y)}{\cancel{f_{X_1}(y)}} dy = \int F_{X_2}(y)\, dy$$

$$= \int_0^t 1 - e^{-\lambda y} dy = \left[y \Big|_0^t \right] - \left[\frac{-e^{-\lambda y}}{\lambda} \Big|_0^t \right]$$

$$= \boxed{t + \frac{e^{-\lambda t}}{\lambda} - \frac{1}{\lambda}}$$

2

6. (6 points) Let $\{X_i, i = 1, ..., r\}$ be r independent identically distributed (iid) random variables with Exp(λ) distribution, and let $Y = \sum_{i=1}^{r} X_i$.

(a) (3 points) Derive the Laplace transform of the PDF of X_i.

(b) (3 points) Derive the Laplace transform of the PDF of Y.

(a) $X_i \sim Exp(\lambda) \rightarrow P_{X_i}(x) = \lambda e^{-\lambda x}$

$$F^*_{X_i}(s) = \int_{0^-}^{\infty} \lambda e^{-\lambda x} e^{-sx}\, dx = \int_0^{\infty} \lambda e^{-\lambda(\lambda+s)x}\, dx = \lambda \left[\frac{e^{-(\lambda+s)x}}{-\lambda+s} \Big|_0^{\infty} \right] = \boxed{\frac{\lambda}{s+\lambda}}$$

(b) $Y = X_1 + X_2 + X_3 + \ldots$

$f_Y(y) = f_{X_1}(x_1) \otimes f_{X_2}(x_2) \otimes \ldots \rightarrow F^*_Y(s) = F^*_{X_1}(s) \cdot F^*_{X_2}(s) \cdot F^*_{X_3}(s) \cdot \ldots$

$$F^*_Y(s) = (F^*_{X_i}(s))^r$$

$$= \left(\frac{\lambda}{s+\lambda}\right)^r = \boxed{\frac{\lambda^r}{(s+\lambda)^r}}$$

6